On free arrangements of lines

Sergey Yuzvinsky

University of Oregon

Spring AMS Meeting, March 2016,
Special Section in Honor of Mike Falk

The Terao problem

Definition

Let $\mathcal{A}$ be a central arrangement of hyperplanes in a linear space $V \simeq \mathbb{C}^r$. We assume the arrangement is essential in which case r is *the rank* of it.

The arrangement is *free* if its $\mathbb{C}$- module of derivations is free. *The Terao problem* (also called conjecture) asks if this property of arrangement is *combinatorial*, i.e., determined by the intersection lattice L of $\mathcal{A}$.

Although that question was asked by Hiroaki Terao around 1980 the complete answer is still unknown even for $r = 3$.

Goal

In this talk we deal with rank 3 arrangements only. One of the goals of this work in progress is to find combinatorial properties of an arrangement, i.e., properties of its intersection lattice necessary for the arrangement to be free.

Notations

Let us introduce some notations. The arrangement poset L is ranked and has the only two non-trivial levels L_1 of rank 1 elements and L_2 of rank 2 elements. We can projectivize and thus call elements from L_1 *lines* and ones from L_2 *points*. We put $|L_1| = n$ and $|L_2| = N$. The set of points L_2 can be partitioned as $L_2 = \bigcup_{i=2}^{M} L^i$ where L^i consists of points of multiplicity i ($i = 2, \ldots, M$). For $X \in L^i$ we write $m(X) = i$ and put $N_i = |L^i|$ for each i.

Sometimes we want to emphasize dependence of $\mathcal{A}$, for example we write $M(\mathcal{A})$.

Poincaré polynomial

Recall that the Poincaré polinomial $\pi = \pi(L, t)$ for rank 3 is $\pi = 1 + nt + a_2 t^2 + a_3 t^3$ with $a_2 = \sum_{X \in L_2} \mu(X)$ where the value of Möbius function μ is $\mu(X) = m(X) - 1$. Then $a_3 = 1 + a_2 - n$. For every arrangement, π factors with one factor $1 + t$ and the other $\pi_0 = 1 + (n-1)t + a_3 t^2$.

Poincaré polynomial

Recall that the Poincaré polinomial $\pi = \pi(L, t)$ for rank 3 is $\pi = 1 + nt + a_2 t^2 + a_3 t^3$ with $a_2 = \sum_{X \in L_2} \mu(X)$ where the value of Möbius function μ is $\mu(X) = m(X) - 1$. Then $a_3 = 1 + a_2 - n$. For every arrangement, π factors with one factor $1 + t$ and the other $\pi_0 = 1 + (n-1)t + a_3 t^2$.
Plugging in previous notations we obtain

$$\pi_0 = 1 + (n-1)t + [-(n-1) + \sum_{k+1}^{M} (k-1)N_k]t^2.$$

We attribute $\pi_0(L)$ to an arrangement with the intersection lattice isomorphic to L.

Factorization theorem

The following theorem was proved by Terao in 1981. For rank 3 it says the following.

Theorem

If an arrangement is free then its polynomial π_0 factors as $\pi_0 = (1 + e_1 t)(1 + e_2 t)$ where both e_i are positive integers. If one adjoins 1 to them one gets the degrees of homogeneous minimal generators of the derivation module.

T-lattices

For brevity, let us call the lattices whose π_0 factors *T-lattices* and the numbers e_i *the exponents of a T-lattice*.

T-lattices

For brevity, let us call the lattices whose π_0 factors *T-lattices* and the numbers e_i *the exponents of a T-lattice*.

If L is a T-lattice then its polynomial π has two real roots whence its discriminant is non-negative. Thus we have

$$(n-1)^2 + 4(n-1) - 4\sum_{k=2}^{M}(k-1)N_k \geq 0$$

or

$$\frac{(n-1)(n+3)}{4} \geq \sum_{k=2}^{M}(k-1)N_k.$$

Lattices L with $M(L) = 3$

Here is the main theorem of the talk.

Theorem

If L is a T-lattice with $M(L) = 3$ then the number of atoms n is not larger than 9.

Proof

Proof.

If $M(L) = 3$ the inequality we had on the previous frame becomes

$$\frac{(n-1)(n+3)}{4} \geq (N_2 + 2N_3)$$

and counting pairs of lines in two different ways we also get

$$\binom{n}{2} = N_2 + 3N_3.$$

Eliminating N_3 from the system by subtracting the inequality from the equality and using the equality again we obtain

$$\frac{(n-1)(n-3)}{4} \leq N_3 \leq \frac{1}{3}\binom{n}{2}$$

$\square$

End of proof

Omitting N_3 from the middle we have

$$\frac{(n-1)(n-3)}{4} \leq \frac{1}{3}\binom{n}{2}$$

whence

$$\frac{n-3}{4} \leq \frac{n}{6}$$

which immediately implies

$$n \leq 9.$$

Examples

Examples. Consider the sequence of reflection arrangements of special monomial type $G(3,3,3)$. Recall that they can be given explicitly in $\mathbb{C}^3$ by the equation

$$(x^3 - y^3)(y^3 - z^3)(x^3 - z^3) = 0.$$

We have $n = 9$ and $N = N_3 = 12$. As all reflection arrangements these are free whence their lattices are T-lattices with $\{e_1, e_2\} = \{4, 4\}$.

Examples

Examples. Consider the sequence of reflection arrangements of special monomial type $G(3,3,3)$. Recall that they can be given explicitly in $\mathbb{C}^3$ by the equation

$$(x^3 - y^3)(y^3 - z^3)(x^3 - z^3) = 0.$$

We have $n = 9$ and $N = N_3 = 12$. As all reflection arrangements these are free whence their lattices are T-lattices with $\{e_1, e_2\} = \{4, 4\}$.

This shows that the upper bound 9 in the Theorem is strict.

Corollary

It is well-known that for small n, at least for $n \leq 11$ the Tearao problem has the positive solution. Thus we have the following corollary.

Corollary

For the class of T-lattices with only double or triple points of intersection the Terao problem has the positive solution.

IF SOMEBODY IN THE ROOM HAS KNOWN IT PLEASE SPEAK UP.

L with $M(L) > 3$

Conjecture The set of all arrangement lattices L with $M(L)$ fixed is finite.

L with $M(L) > 3$

Conjecture The set of all arrangement lattices L with $M(L)$ fixed is finite.

The conjecture is open for every $M(L) > 3$. Take for instance, $M(L) = 4$. Then there are the equality and inequality

$$N_2 + 3N_3 + 6N_4 = \binom{n}{2}$$

and

$$N_2 + 2N_3 + 3N_4 \leq \frac{(n-1)(n+3)}{4}.$$

Corollaries

It is not hard to get some corollary from this system. Here are several of them

$$N_4 \geq \frac{(n-1)(n-9)}{12},$$

$$N_3 + 3N_4 \geq \frac{(n-1)(n-3)}{4},$$

$$N_2 + N_3 \leq \frac{3}{2}(n-1).$$

However these inequalities does not forbid the existence of an infinite set of values of n.

Direction to go

In order to use more properties of L it is reasonable to start with the formality property à la Mike Falk and Richard Randell (1986). Shortly the property says that all linear relations among hyperplanes are generated by 3-relations.

Direction to go

In order to use more properties of L it is reasonable to start with the formality property à la Mike Falk and Richard Randell (1986). Shortly the property says that all linear relations among hyperplanes are generated by 3-relations.

It was proved in 1993 that the formality is necessary for the freeness. Although the formality is not combinatorial it together with irreducibility of $\mathcal{A}$ implies the combinatorial property that $\bigcup_{X\in L_2, X\notin L^2}\mathcal{A}_X=\mathcal{A}$. This may be useful for future developments.

SORRY FOR THE ELEMENTARY TALK