Business Calculus
Math 1431

Unit 2.6

The Derivative

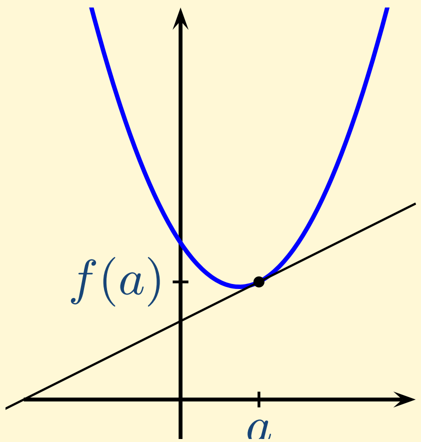
Mathematics Department

Louisiana State University

Introduction

In an earlier lecture we considered the idea of finding the average rate of change of some quantity $Q(t)$ over a time interval $[t_1, t_2]$ and asked what would happen when the time interval became smaller and smaller. This led to the notion of limits, which we have examined in the previous two lectures. In this section we return to our study of average rates of change and apply what we learned about limits. This leads directly to the **derivative.** It turns out that the derivative has an equivalent (purely) mathematical formulation: that of finding the line tangent to a curve at some point. This is where we will start.

Suppose $y = f(x)$ is some given function and a is some fixed point in the domain. We will explore the idea of finding the equation of the line tangent to the graph of f at the point $(a, f(a))$. Look at the picture below:



The **tangent line** at $x = a$ is the unique line that goes through $(a, f(a))$ and just touches the graph there.

The Slope?

Recall the an equation of the line is determined once we know a point P and the slope m. If $P = (a, f(a))$ then the equation of the line tangent to the curve will take the form

$$y - f(a) = m(x - a)$$

for some slope m that we have to determine.

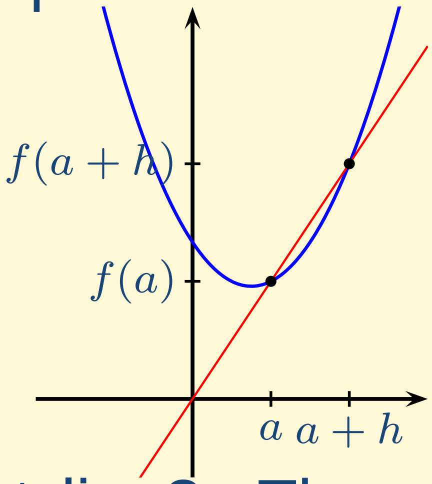
How do we determine the slope of the tangent line?

We will do so by a limiting process.

Secant lines

Given the graph of a function $y = f(x)$ we call a **secant line** a line that connects two points on a graph.

In the graph to the right a red line joins the points $(a, f(a))$ and $(a+h, f(a+h)$. (Think of h as a relatively small number.)

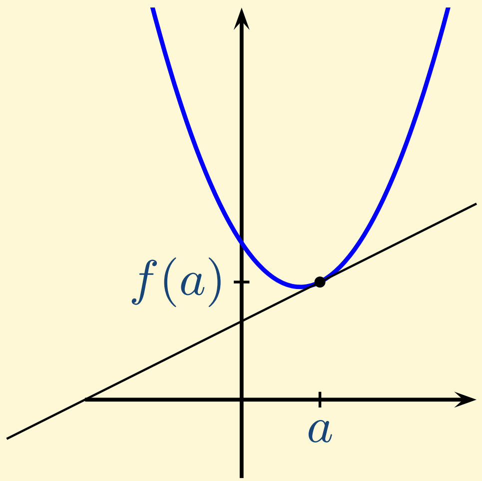


Now, what is the slope of the secant line? The change in y is $\Delta y = f(a+h) - f(a)$ and the change in x is $\Delta x = a + h - a = h$. So the slope of the secant line is:

$$\frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{h}.$$

Small h

The following graph illustrates what happens when we choose smaller values of h. Notice how the secant lines get closer to the tangent line.



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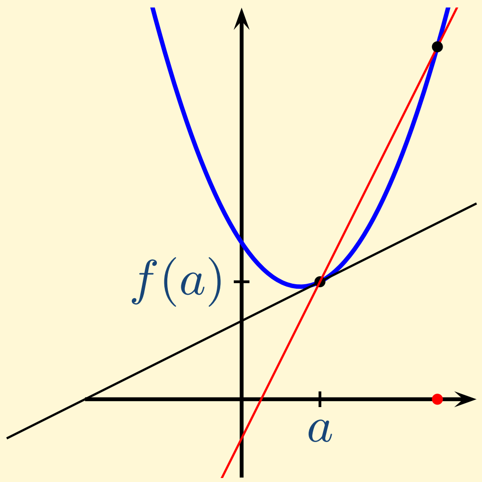

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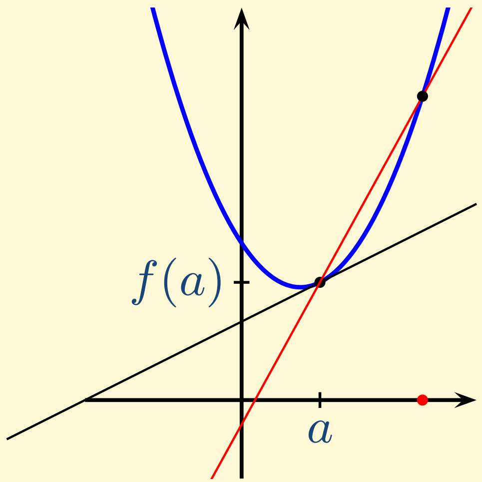

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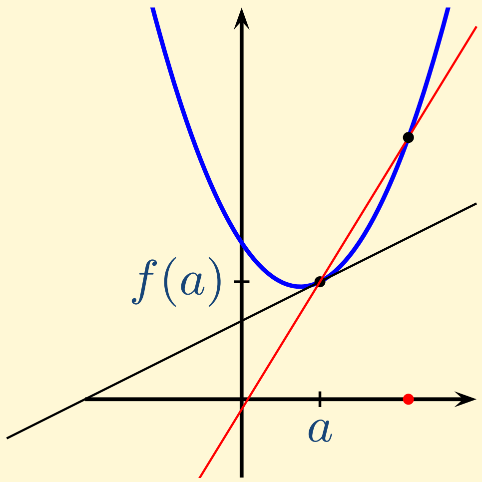

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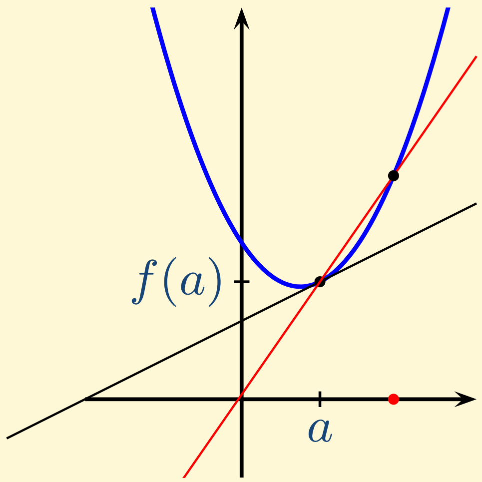

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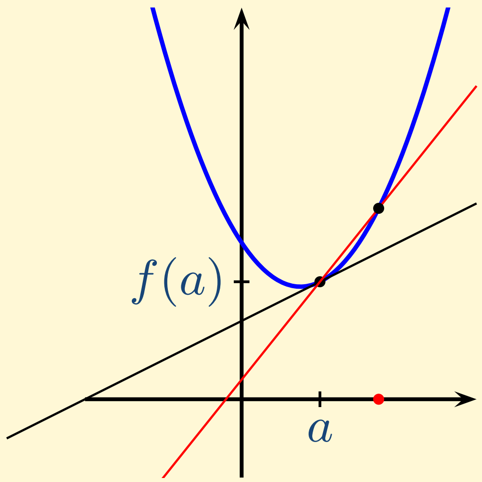

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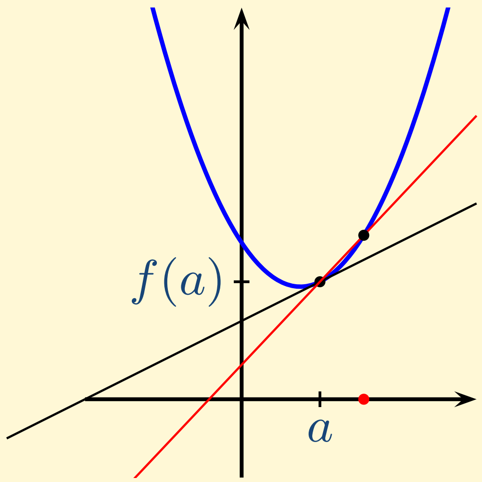

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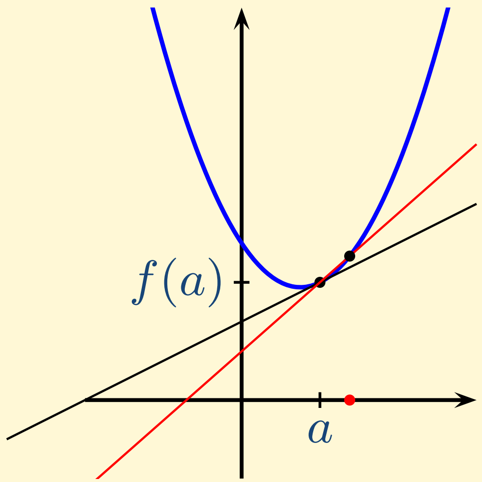

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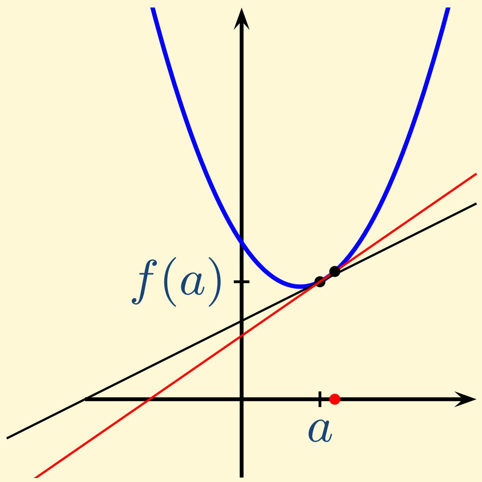

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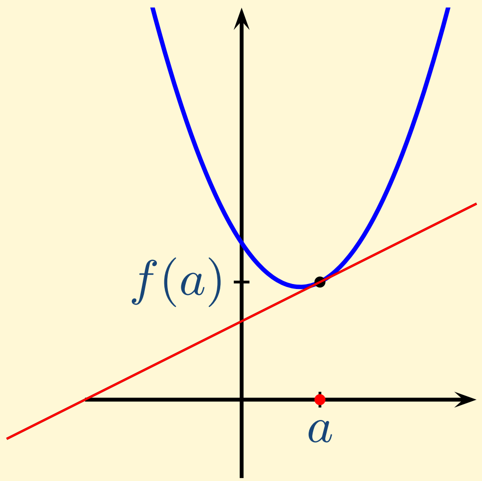

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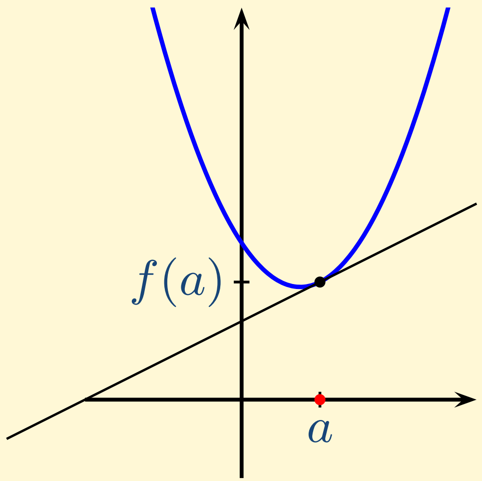

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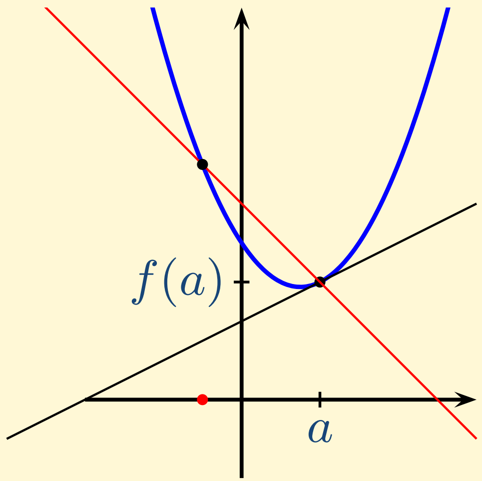

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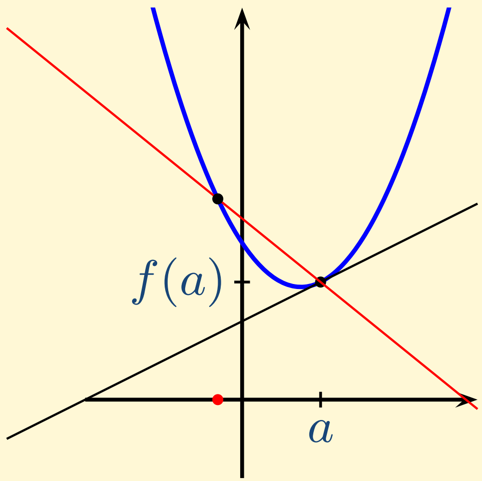

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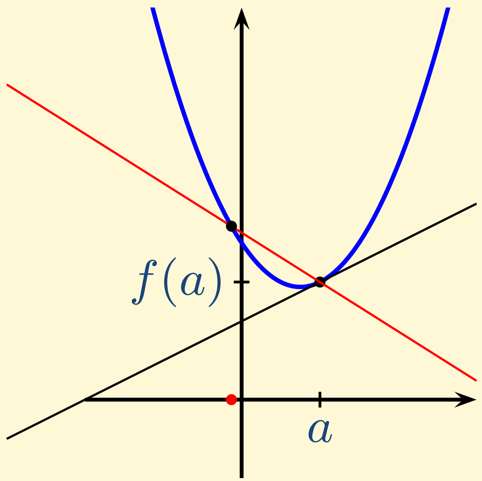

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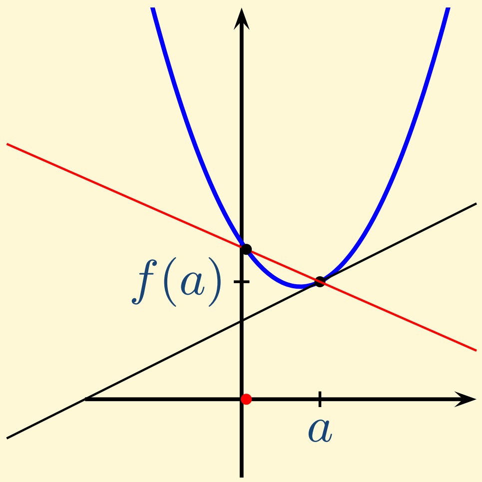

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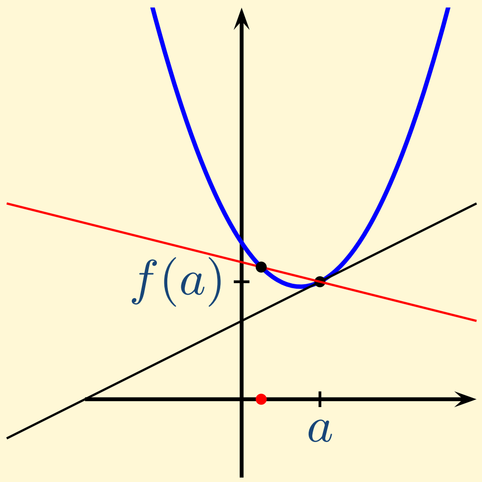

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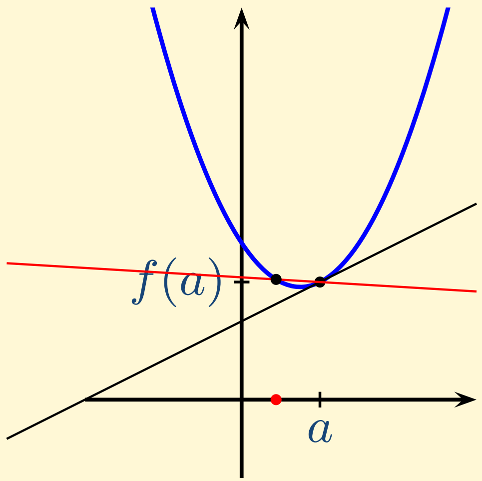

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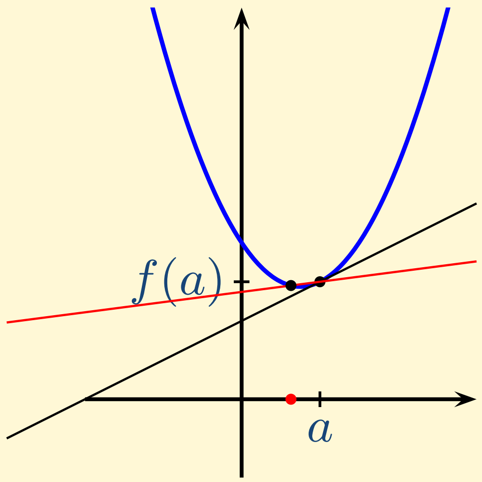

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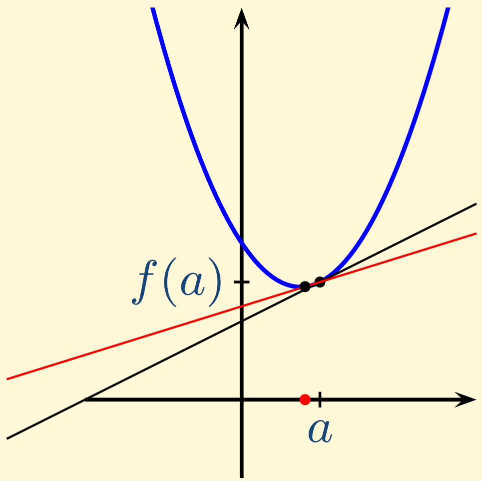

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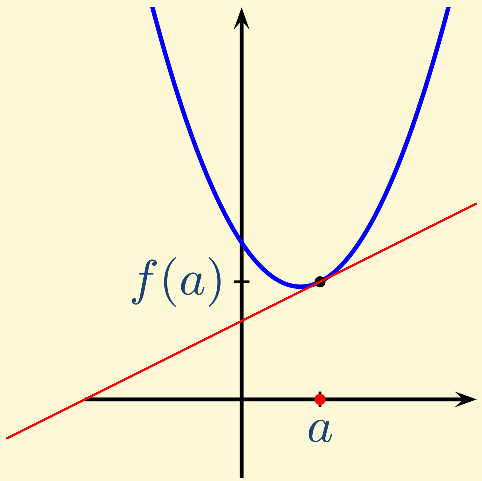

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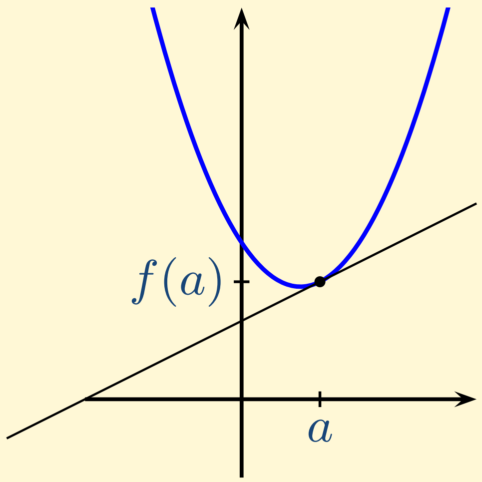

The main idea

The main idea is to let h get smaller and smaller. Remember all we need is the slope of the tangent line so we will compute

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

In the following animation notice how the slopes of the secant lines approach the slope of the tangent line.

Convergence of Secant Lines: An animation

Notice how the secant line and hence its slopes converge to the tangent line.

Slope of the Tangent Line

Since the slope of the secant lines are given by the formula

$$\frac{f(a+h)-f(a)}{h},$$

the slope of the tangent line is given by

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h},$$

when this exists. We will call this number the **derivative of f at a** and denote it $f'(a)$. Thus

The derivative of f at $x=a$ is given by

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

To illustrate some of these ideas let's consider the following example:

Example 1: Let $f(x) = x^2$ and fix $a = 1$. Compute the slope of the secant line that connects $(a, f(a))$ and $(a + h, f(a + h)$ for

- $h = 1$
- $h = .5$
- $h = .1$
- $h = .01$

Compute the derivative of f at $x = 1$, i.e. $f'(1)$. Finally, find the equation of the line tangent to the graph of $f(x) = x^2$ and $x = 1$.

Let m_h be the slope of the secant line. Then

$$m_h = \frac{f(1+h) - f(1)}{h} = \frac{(1+h)^2 - 1}{h}.$$

- For $h = 1$ $m_1 = \frac{(1+1)^2-1}{1} = 4 - 1 = 3.$
- For $h = .5$ $m_{.5} = \frac{(1.5)^2-1}{.5} = 2.5.$
- For $h = .1$ $m_{.1} = \frac{(1.1)^2-1}{.1} = 2.1$
- For $h = .01$ $m_{.01} = \frac{(1.01)^2-a}{.01} = 2.01$

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- For $h = .01$ $m_{.01} = \frac{(1.01)^2-a}{.01} = 2.01$

What do you think the limit will be as h goes to 0?

The limit is the derivative. We compute

$$
\begin{aligned}
f'(1) &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \\
&= \lim_{h\to 0} \frac{(1+h)^2-1}{h} \\
&= \lim_{h\to 0} \frac{1+2h+h^2-1}{h} \\
&= \lim_{h\to 0} \frac{2h+h^2}{h} \\
&= \lim_{h\to 0} 2+h = 2.
\end{aligned}
$$

Therefore the slope of the tangent line is $m = 2$.

The equation of the tangent line is now an easy matter. The slope m is 2 and the point P that the line goes through is $(1, f(1)) = (1, 1)$. Thus, we get

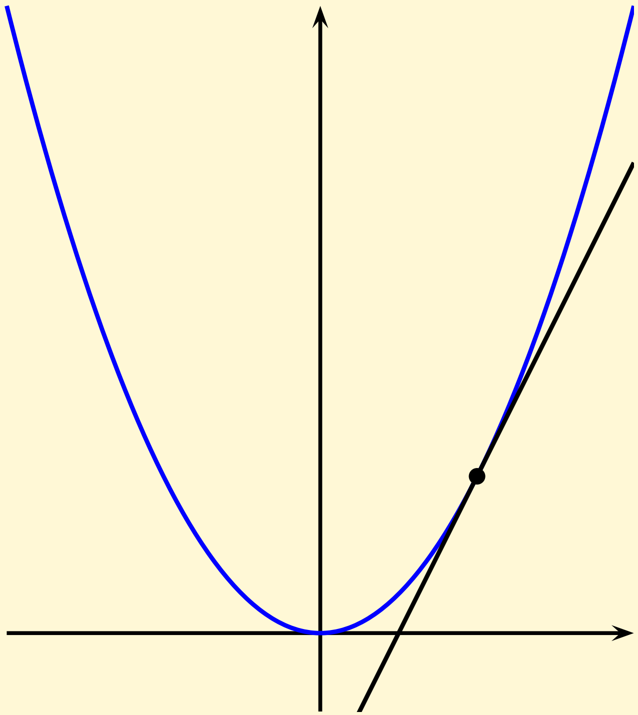
$$y - 1 = 2(x - 1)$$

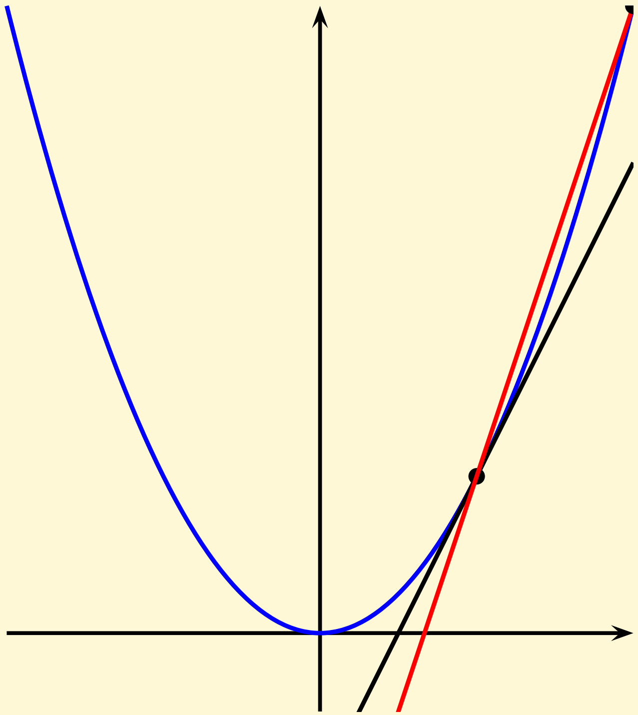
or

$$y = 2x - 1.$$

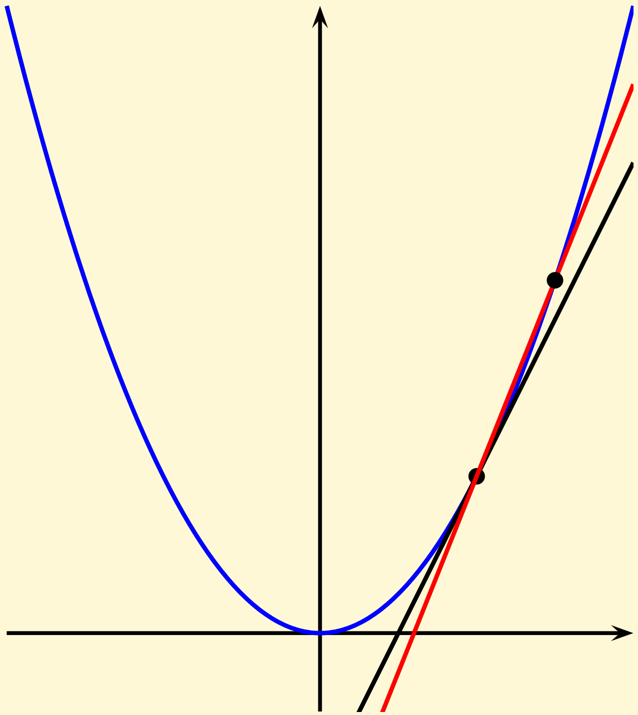
On the next slide we give a graphical representation of what we have just computed.

$f(x) = x^2$

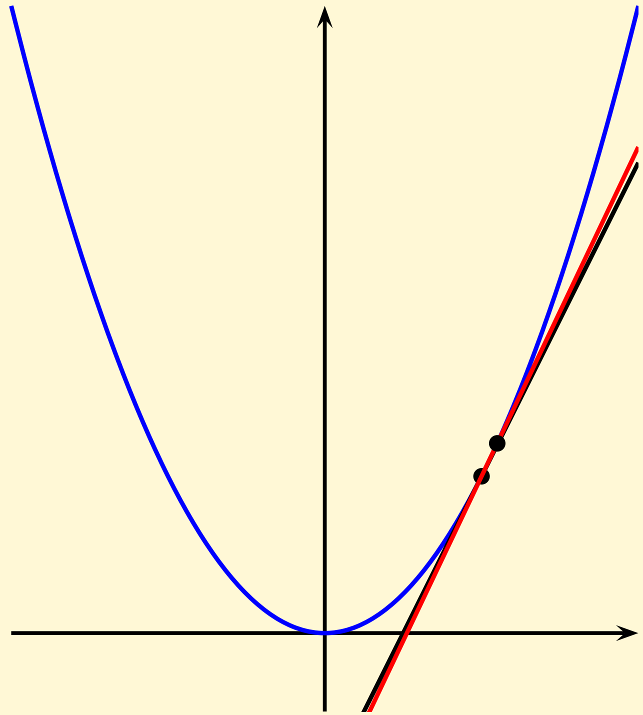




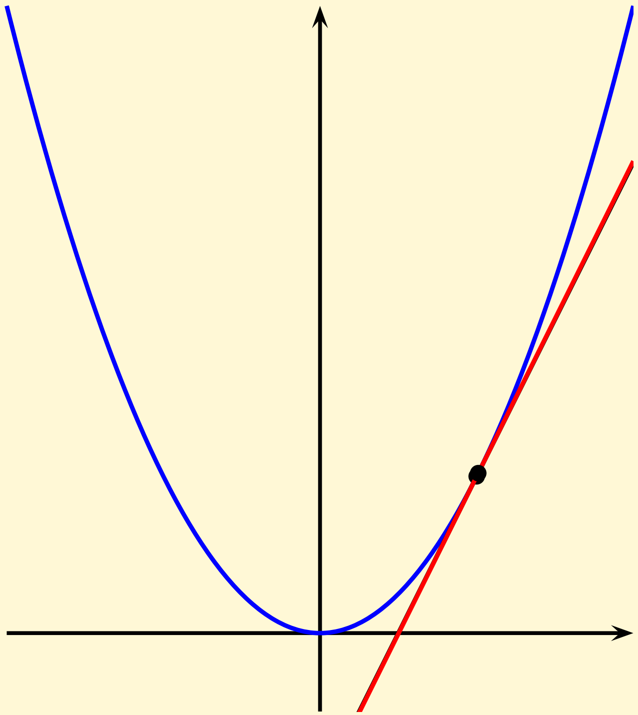
Here $h = 1$ and the secant line goes through the points $(1, 1)$ and $(2, 4)$



Here $h = .5$ and the secant line goes through the points $(1, 1)$ and $(1.5, 2.25)$



Here $h = .1$ and the secant line goes through the points $(1, 1)$ and $(1.1, 1.21)$



Here $h = .01$ and the secant line goes through the points $(1, 1)$ and $(1.01, 1.0201)$

Rates of Change

Rates of change

Given a function $y = f(x)$ the **difference quotient**

$$\frac{f(x+h) - f(x)}{h}$$

measures the **average rate of change of** y **with respect to** x over the interval $[x, x+h]$. As the interval becomes smaller, i.e. as h goes to 0, we obtain the **instantaneous rate of change**

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

which is precisely our definition of the derivative $f'(x)$. Thus, the derivative measures in an instant the rate of change of $f(x)$ with respect to x.

Velocity

If $s(t)$ is the distance travelled by an object (your car for instance) as a function of time t then the quantity

$$\frac{s(t+h)-s(t)}{h}$$

is the average rate of change of distance over the time interval $[t, t+h]$. This is none other than your average velocity. The quantity

$$\lim_{h\to 0}\frac{s(t+h)-s(t)}{h}$$

is the instantaneous rate of change: This is none other than your velocity which you would read from your speedometer. Thus your speedometer can be thought of as a derivative machine.

Example

Example 2: Suppose the distance travelled by a car (in feet) is given by the function $s(t) = \frac{1}{2}t^2 + t$ where $0 \leq t \leq 20$ is measured in seconds.

- Find the average velocity over the time interval
  - $[10, 11]$
  - $[10, 10.1]$
  - $[10, 10.01]$
- Find the instantaneous velocity at $t = 10$.
- Compare the above results.

- The average velocity over the given time intervals are:
  - $\frac{s(11)-s(10)}{11-10} = \frac{1}{2}(11)^2 + 11 - (\frac{1}{2}(10)^2 + 10) = 11.5$ (ft/sec)
  - $\frac{s(10.1)-s(10)}{10.1-10} = \frac{1.105}{.1} = 11.05$ (ft/sec)
  - $\frac{s(10.01)-s(10)}{10.01-10} = \frac{1.1005}{.01} = 11.005$ (ft/sec)

- We could probably guess that the instantaneous velocity at $t = 10$ is 11 (ft/sec). But lets calculate this using the definition

$$\begin{aligned}
s'(t) &= \lim_{h\to 0} \frac{s(t+h) - s(t)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{2}(t+h)^2 + (t+h) - (\frac{1}{2}t^2 + t)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{2}(t^2 + 2th + h^2) + t + h - \frac{1}{2}t^2 - t}{h} \\
&= \lim_{h\to 0} \frac{th + \frac{1}{2}h^2 + h}{h} \\
&= \lim_{h\to 0} t + \frac{1}{2}h + 1 = t + 1.
\end{aligned}$$

Notice that we have calculated the derivative at any point t:

$$s'(t) = t + 1.$$

We now evaluate at $t = 10$ to get

$$s'(10) = 11$$

just as we expected.

The average velocity over the time intervals $[10, 10 + h]$ for $h = 1$, $h = .1$ and $h = .01$ become closer to the instantaneous velocity at $t = 10$. This is as we should expect.

Finding the derivative of a function using the definition

Notation

Differential calculus has various ways of denoting the derivative, each with their own advantages. We have used the **prime notation**, $f'(x)$ (read: "f prime of x"), to denote the derivative of $y = f(x)$. You will also see y' written when it is clear $y = f(x)$. The prime notation is simple, quick to write, but not very inspiring.

Another notation is

$$\frac{df}{dx} \quad \text{or} \quad \frac{dy}{dx}.$$

This notation is much more suggestive. Recall that the derivative is the limit of the difference quotient $\frac{\Delta y}{\Delta x}$: the change in y over the change in x. The notation "dy" or "df" is used to suggest the instantaneous change in y after the limit is taken and likewise for dx. One must not read too much into this notation. $\frac{df}{dx}$ is **not** a fraction but the limit of a fraction.

There are other notations that are in use but these are the two most common.

An outline

To compute the derivative $\frac{df}{dx} = f'(x)$ of a function $y = f(x)$ using the definition follow the steps:

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1. Find the change in y: $f(x+h) - f(x)$
2. Compute $\frac{f(x+h)-f(x)}{h}$
3. Determine $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

Example

Example 3: Find the derivative of $y = x^3 - x$.

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Let $f(x) = x^3 - x$. Then

$$\begin{aligned} f(x+h) - f(x) &= (x+h)^3 - (x+h) - (x^3 - x) \\ &= x^3 + 3x^2h + 3xh^2 + h^3 \\ &\quad -x - h - x^3 + x \\ &= 3x^2h + 3xh^2 + h^3 - h \end{aligned}$$

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Next we get

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{3x^2h + 3xh^2 + h^3 - h}{h} \\ &= 3x^2 + 3xh + h^2 - 1 \end{aligned}$$

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Next we get

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Finally, $\frac{dy}{dx} = \lim_{h \to 0} 3x^2 + 3xh + h^2 - 1 = 3x^2 - 1$.

Example

Example 4: Find the equation of the line tangent to

$$f(x) = \sqrt{x}$$

at the point $(4, 2)$.

We need the slope of the tangent line at this point. This is $f'(4)$.

$$\begin{aligned}
f'(4) &= \lim_{h\to 0} \frac{f(4+h)-f(4)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{4+h}-2}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{4+h}-2}{h}\,\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2} \\
&= \lim_{h\to 0} \frac{4+h-4}{h(\sqrt{4+h}+2)} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{4+h}+2} = \frac{1}{4}.
\end{aligned}$$

$f'(4) = \frac{1}{4}$ **and** $P = (4, 2)$

Given a point and a slope we compute the line:

$$y - 2 = \frac{1}{4}(x - 4)$$

or

$$y = \frac{1}{4}x + 1.$$

Do not despair!

Admittedly, the calculation of a derivative using the definition can be tedious. However, in the next chapter we will discuss a set of rules for differentiation that will allow us to calculate the derivative of many commonly encountered functions very easily. Nevertheless, it is important that you understand the definition and the underlying meaning of the derivative; at times, it will be necessary to come back to it.

Differentiation and Continuity

Reformulation of Continuity

In the last section we discussed the meaning of continuity. Recall a function $y = f(x)$ is continuous at a point a if $f(a)$ is defined and

$$\lim_{x \to a} f(x) = f(a).$$

If we let $x = a + h$ then x approaches a if h approaches 0. This observations allows us the give an equivalent definition for continuity: $f(a)$ is defined and

$$\lim_{h \to 0} f(a + h) - f(a) = 0.$$

Differentiable functions are Continuous

A function is said to be **differentiable at a point** $x = a$ if $f'(a)$ exists. This means that the limit $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists. We say f is **differentiable on an interval** (a, b) if it is differentiable at every point in the interval.

Notice the next theorem:

Theorem: A function that is differentiable at a point $x = a$ is continuous there.

We have not been proving many theorems but this one is easy and short enough that we will do so on the next slide.

Proof: To say f is differentiable at $x = a$ means

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists and is a finite number, denoted $f'(a)$. Thus

$$\begin{aligned} \lim_{h \to 0} (f(a+h) - f(a)) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \cdot h \\ &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \cdot \lim_{h \to 0} (h) \\ &= f'(a) \cdot 0 = 0. \end{aligned}$$

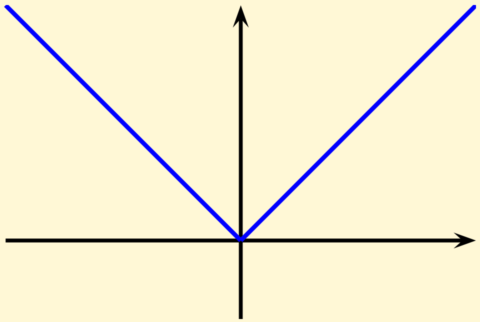
This means that f is continuous at $x = a$.

Continuity does not imply Differentiability

We must not read something that is not in this theorem. Though a differentiable function is necessarily continuous a continuous function is not necessarily differentiable. Consider this classic example:

$$y = |x|\,.$$

At $x = 0$ there are several lines that just touch the graph at $(0, 0)$; it is not unique.



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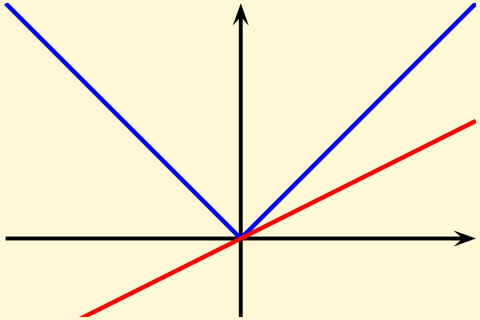

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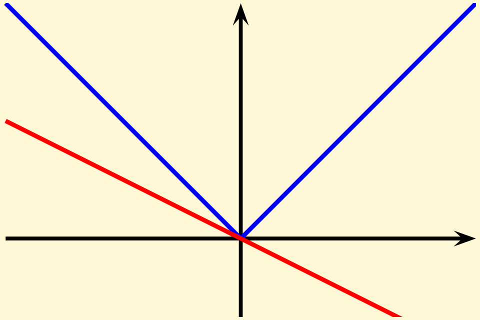

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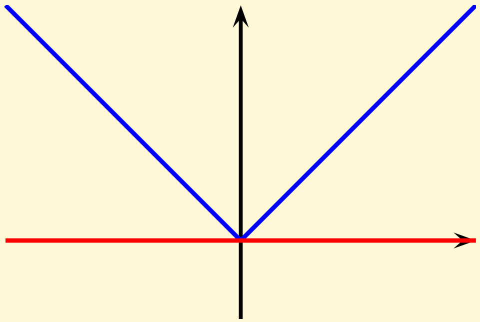

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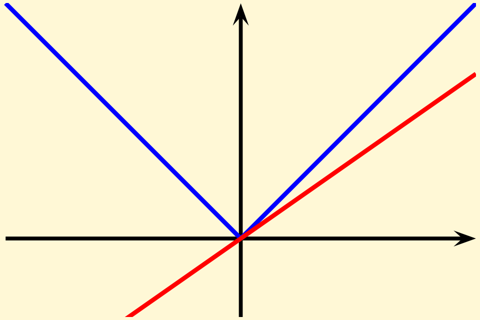

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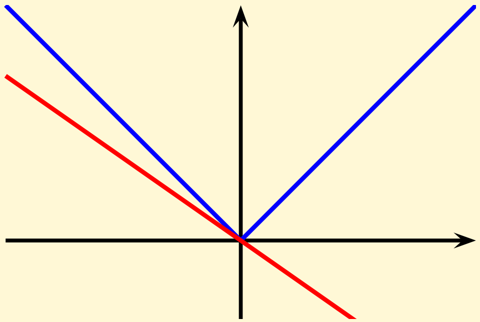

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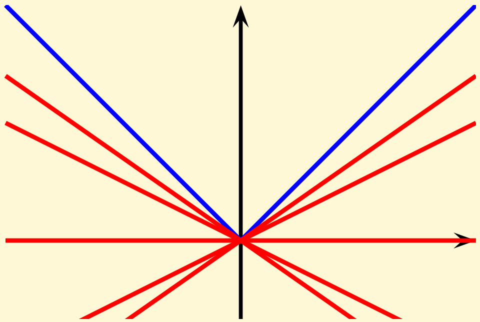

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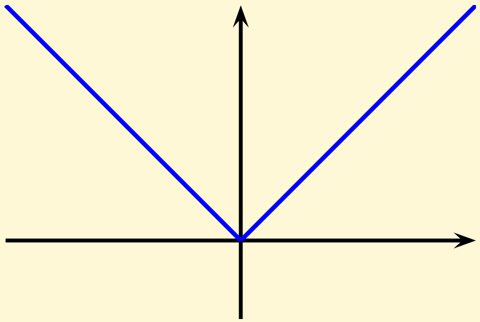

Remember, tangent lines are **unique** and since $y = |x|$ has no unique tangent line it is not differentiable at $x = 0$.

Consider what happens here in terms of the definition:

$$y'(0) = \lim_{h\to 0} \frac{|0+h| - 0}{h} = \lim_{h\to 0} \frac{|h|}{h}.$$

Now, to compute this limit we will consider the left and right-hand limits.

If h is positive then $|h| = h$ and

$$y'(0) = \lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0} \frac{h}{h} = 1.$$

If h is negative then $|h| = -h$ and

$$y'(0) = \lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0} \frac{-h}{h} = -1.$$

The left and right hand limits are not equal therefore $\lim_{h\to 0} \frac{|h|}{h}$ does not exist.

If $y = |x|$ then y is continuous but not differentiable at $x = 0$.

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- Notation: y' or $\frac{dy}{dx}$.

In-Class Exercises

In-Class Exercise

In-Class Exercise 1: At a fixed temperature the volume V (in liters) of 1.33 g of a certain gas is related to its pressure p (in atmospheres) by the formula

$$V(p) = \frac{1}{p}.$$

What is the average rate of change of V with respect to p as p increases from 5 to 6 ?

1. 1
2. $\frac{1}{6}$
3. $-\frac{1}{5}$
4. $-\frac{1}{30}$
5. $-\frac{1}{6}$

return

In-Class Exercise 2: Use the definition of the derivative to find y' if

$$y = 4x^2 - x.$$

1. $4x^2 - 1$
2. $8x - 1$
3. $8x$
4. $4x^2 - x$
5. None of the above

return

In-Class Exercise 3: Find the equation of the line tangent to

$$y = x^2 + x$$

at the point $(1, 2)$.

1. $y = 3x - 1$
2. $y = 3x - 5$
3. $y = 2x$
4. $y = 2x - 3$
5. None of the above