

Problem 1. Limit of Convex functions

1. Let $\{\phi_\alpha\}$ be a family of convex functions on (a, b) and

$$\phi = \sup_{\alpha} \{\phi_\alpha\}.$$

Prove that ϕ is convex on (a, b) if it is finite.

2. Let ψ_n be a sequence of convex functions on (a, b) and

$$\psi(x) = \lim_{n \rightarrow \infty} \psi_n(x) \quad \forall x \in (a, b).$$

Prove that ψ is convex on (a, b) .

3. Let f_n be a sequence of convex functions on (a, b) . Prove that $\overline{\lim} f_n$ is convex on (a, b) .
4. Find a counterexample to show that minimum of two convex functions may not be convex.

Problem 2. Logarithm Function If ϕ is convex on (a, b) and if ψ is convex and nondecreasing on the range of ϕ .

1. Prove that $\psi \circ \phi$ is convex on (a, b) .
2. For $\phi > 0$, show that the convexity of $\log \phi$ implies the convexity of ϕ , but not vice versa.

Problem 3. Mid-point Convex Assume that ϕ is a continuous real function on (a, b) such that

$$\phi\left(\frac{x+y}{2}\right) \leq \frac{1}{2}\phi(x) + \frac{1}{2}\phi(y), \quad \forall x \in (a, b), \forall y \in (a, b).$$

Prove that ϕ is convex.

Problem 4. Discrete Jensen's Inequality Define $\{x_i\}_1^n$ is a finite set of points and $\alpha_i > 0, i = 1, \dots, n$, is $\sum_{i=1}^n \alpha_i = 1$.

1. Let ϕ be a convex function. Show that

$$\phi\left(\sum_{i=1}^n \alpha_i x_i\right) \leq \sum_{i=1}^n \alpha_i \phi(x_i).$$

2. Show that if $x_i > 0, i = 1, \dots, n$, then

$$\prod_{i=1}^n x_i^{\alpha_i} \leq \sum_{i=1}^n \alpha_i x_i.$$

Problem 5. An application of Jensen's Inequality Let f be a positive measurable function on $[0, 1]$. Show that

$$\int_0^1 f(s) ds \int_0^1 \log f(t) dt \leq \int_0^1 f(x) \log f(x) dx.$$

Problem 6. Application of Hölders inequality

1. Show that if $k_1, k_2, \dots, k_n > 1$ and $\sum_{i=1}^n \frac{1}{k_i} = 1$, then if $f_i \in L^{k_i}$ for each i ,

$$\int |f_1 f_2 \cdots f_n| dx \leq \left(\int |f_1|^{k_1} dx \right)^{\frac{1}{k_1}} \cdots \left(\int |f_n|^{k_n} dx \right)^{\frac{1}{k_n}}.$$

2. Show that if $\alpha, \beta, \gamma > 0$ and $p < \frac{1}{(\alpha+\beta+\gamma)}$, then

$$\int_0^2 \frac{1}{(x^\alpha |x-1|^\beta |x-2|^\gamma)^p} dx < \infty.$$

3. Show that if $f \in L^p$ and $g \in L^q$ where $p \geq 1$ and $q \geq 1$, then $fg \in L^r$ for r such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.

Problem 7. Application of two inequalities Let $f_n \in L^2(a, b), n = 1, 2, \dots$, let $f \in L^2(a, b)$ and let $\lim_{n \rightarrow \infty} \|f_n - f\|_2 = 0$. Show that

$$1. \int_a^b f^2 dx = \lim_{n \rightarrow \infty} \int_a^b f_n^2 dx.$$

$$2. \text{ If } a \text{ and } b \text{ are finite, then } \int_a^t f dx = \lim_{n \rightarrow \infty} \int_a^t f_n dx, \quad a \leq t \leq b.$$

Problem 8. Essentially bounded function Let $f : X \rightarrow [0, \infty)$ be a measurable, essentially bounded function and $m(X) < \infty$.

1. Show that

$$\lim_{n \rightarrow \infty} \left(\int f^n dx \right)^{\frac{1}{n}} = \text{ess sup } f.$$

2. Show that

$$\lim_{n \rightarrow \infty} \frac{\int f^{n+1} dx}{\int f^n dx} = \text{ess sup } f.$$

Problem 9. Application Suppose $m(\Omega) = 1$ and suppose f and g are positive measurable functions on Ω such that $fg \geq 1$. Show that

$$\int_{\Omega} f dx \int_{\Omega} g dx \geq 1.$$

Problem 10. Application Suppose $f_n \in L^p$, for $n = 1, 2, 3, \dots$, and $\|f_n - f\|_p \rightarrow 0$ and $f_n \rightarrow g$ a.e. as $n \rightarrow \infty$. What relation exists between f and g if $p \geq 1$?