

Problem Set 1 : Integration by parts

a) Let h be integrable on $[a, b]$ and k be a measurable function which is bounded a.e. on $[a, b]$. Show that hk is integrable on $[a, b]$.

b) Show that the product of two absolutely continuous functions is absolutely continuous.

c) Suppose F is absolutely continuous and g is integrable on $[a, b]$, Define $f(x) = F'(x)$ a.e. on $[a, b]$ and

$$G(x) = \int_a^x g(t) dt + C, \quad \text{where } C \text{ is constant.}$$

i) Show that Fg and fG are integrable on $[a, b]$.

ii) Show that $\int_a^b F(t)g(t) dt + \int_a^b f(t)G(t) dt = F(b)G(b) - F(a)G(a)$.

Problem Set 2 : Series

Let $\{f_k\}$ be a sequence of monotone increasing functions on $[a, b]$. Assume that the series $s(x) = \sum_{k=1}^{\infty} f_k(x)$ converges on $[a, b]$. Show That

$$s'(x) = \sum f'_k(x) \quad \text{a.e. in } [a, b]$$

a) Let $s_m = \sum_{k=1}^m f_k(x)$ and $r_m = \sum_{k=m+1}^{\infty} f_k(x)$. Show that s_m and r_m are monotone increasing and $s = s_m + r_m$.

b) Show that $s' = s'_m + r'_m$ a.e. and $\sum_{k=1}^{\infty} f'_k(x) \leq s'$ a.e.

c) Show that $r'_m \rightarrow 0$ a.e. as $m \rightarrow \infty$ using the following subproblems.

c1) Show that there exists a subsequence m_j satisfying $\sum r_{m_j}(x)$ converges at both $x = a$ and $x = b$. (we note that $a_n \rightarrow 0$ does not imply $\sum a_n < \infty$ but we can choose n_k such that $\sum a_{n_k} < \infty$.)

c2) Show that $\int_a^b \sum r'_{m_j} < \infty$

c3) Explain why this implies $r'_m \rightarrow 0$ a.e.

Problem Sets 3 : Using Problem sets 2 Turn in on Friday, Dec 5.

Let ϕ be the cantor function. Set $\phi(x) = 0$ for $x < 0$ and $\phi(x) = 1$ for $x > 1$. Let $\{[a_n, b_n]\}$ be the sequence of closed intervals $[0, 1]$, $[0, \frac{1}{2}]$, $[\frac{1}{2}, 1]$, $[0, \frac{1}{4}]$, $[\frac{1}{4}, \frac{1}{2}]$, \dots , $[0, \frac{1}{8}]$, $[\frac{1}{8}, \frac{1}{4}]$, \dots and define

$$\phi_n = \phi\left(\frac{x - a_n}{b_n - a_n}\right) \quad \text{and} \quad f(x) = \sum_{k=1}^{\infty} 2^{-k} \phi_k(x).$$

a) Show that f is continuous and strictly increasing on $[0, 1]$

b) Show that $f' = 0$ a.e.