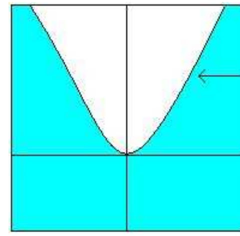
**1.** Sketch the domain of $f(x,y)=\ln(x^2-y)$. Indicate if the boundary is part of the domain.




The domain of the function $g(u)=\ln u$ is $u>0$. So the domain of the function $f(x,y)=\ln(x^2-y)$ is $x^2-y>0$, which is $y<x^2$.
The graph $y=x^2$ is a parabola. So the region described by $y<x^2$ is the part below this parabola. The boundary $y=x^2$ is not part of the domain $y<x^2$.

**2.** Find $\displaystyle\lim_{(x,y)\to(0,0)}\left(\frac{1}{xy}-\frac{1}{xy(x+1)}\right)$ if it exists, or show that the limit does not exist.

First, $\displaystyle\lim_{(x,y)\to(0,0)}\left(\frac{1}{xy}-\frac{1}{xy(x+1)}\right)=\lim_{(x,y)\to(0,0)}\frac{1}{xy}\left(1-\frac{1}{x+1}\right)=\lim_{(x,y)\to(0,0)}\frac{1}{xy}\cdot\frac{x}{x+1}=\lim_{(x,y)\to(0,0)}\frac{1}{y(x+1)}$.

Let us consider the line $L$ given by $y=x$. Along this line the function becomes $1/x(x+1)$. Now,

$$\text{Limit along line } L \text{ from the positive side} = \lim_{x\to 0^+}\frac{1}{x(x+1)}=+\infty$$
$$\text{Limit along line } L \text{ from the negative side} = \lim_{x\to 0^-}\frac{1}{x(x+1)}=-\infty$$

These two limits are different and hence $\displaystyle\lim_{(x,y)\to(0,0)}\left(\frac{1}{xy}-\frac{1}{xy(x+1)}\right)$ does not exist.

**3.** Find $f_{yxy}$ for $f(x,y)=x+xe^{-x}y+e^{-x}y^2$.

First, $f_y=xe^{-x}+2e^{-x}y$.
Then, $f_{yy}=2e^{-x}$.
So, $f_{yxy}=f_{yyx}=-2e^{-x}$.

**4.** Estimate the value of $(1.02)^5(1.99)^3$ by using linear approximation.

Let $f(x,y)=x^5y^3$. Then $f_x=5x^4y^3$ and $f_y=3x^5y^2$. Therefore $f_x(1,2)=40$ and $f_y(1,2)=12$. It follows that

$$f(x,y)\approx f(1,2)+f_x(1,2)(x-1)+f_y(1,2)(y-2)=8+40(x-1)+12(y-2).$$

Now $f(1.02)\approx 8+40\times 0.02+12\times(-0.01)=8+0.8-0.12=8.68$.

**5.** Find an equation of the plane tangent to the surface $x^2+z^2e^{y-x}=13$ at point $P=(2,2,3)$.

In general, the tangent plane for $f(x,y,z)=0$ at $(a,b,c)$ is given by $\nabla f(a,b,c)\cdot(x-a,y-b,z-c)=0$.

In this case, $f_x=2x-z^2e^{y-x}$ implying $f_x(2,2,3)=4-9=-5$;
$f_y=z^2e^{y-x}$ implying $f_y(2,2,3)=9$;
$f_z=2ze^{y-x}$ implying $f_z(2,2,3)=6$.

Therefore, the tangent plane is $-5(x-2)+9(y-2)+6(z-3)=0$ or $5x-9y-6z+26=0$.

**6.** Find the directional derivative of $f(x,y,z)=\sqrt{xyz}$ at $(2,4,2)$ in the direction $\mathbf{v} =< 4,\ 2,\ -4 >$.

First, $f_x = \dfrac{1}{2\sqrt{x}}\sqrt{yz},\quad f_y = \dfrac{1}{2\sqrt{y}}\sqrt{xz},$ and $f_z = \dfrac{1}{2\sqrt{z}}\sqrt{xy}$, so $\nabla f(2,4,2) = <1,\ 1/2,\ 1>$.

Next, $||\mathbf{v}|| = \sqrt{16+4+16} = \sqrt{36} = 6$, so $\mathbf{v}/||\mathbf{v}|| = <2/3,\ 1/3,\ -2/3>$.

Finally, directional derivative $= \nabla f(2,4,2)\cdot \mathbf{v}/||\mathbf{v}|| = <1,\ 1/2,\ 1> \cdot <2/3,\ 1/3,\ -2/3> = 1/6$.

**7.** Find $\dfrac{\partial f}{\partial t}$ if $f(x,y) = x+y-xy+x^2-y^2$, and $x = 2s-3t$, $y = 4s+t$. Express your answer in terms of $s,t$ and also simplify it.

$$\frac{\partial f}{\partial x} = 1-y+2x,\quad \frac{\partial f}{\partial y} = 1-x-2y,\quad \frac{\partial x}{\partial t} = -3,\ \text{and}\ \frac{\partial y}{\partial t} = 1.$$

So $\dfrac{\partial f}{\partial t} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial t} = (1-y+2x)(-3)+(1-x-2y)(1) = -2-7x+y = -2-14s+21t+4s+t = -2-10s+22t.$

**8.** Find all critical points of $f(x,y) = x-y^2-\ln(x+y)$. Then determine if they are local maxima, local minima, or saddle points.

First, $f_x = 1-\frac{1}{x+y}$ and $f_y = -2y-\frac{1}{x+y}$. From $f_x = f_y = 0$ we get $1 = \frac{1}{x+y} = -2y$, which implies $y = -1/2$.
Then we deduce from $1 = \frac{1}{x+y}$ that $x = 1-y = 3/2$. So $(3/2,-1/2)$ is the only critical point.

Next, $f_{xx} = (x+y)^{-2}$, $f_{yy} = -2+(x+y)^{-2}$, and $f_{xy} = (x+y)^{-2}$.
It follows that $D(3/2,-1/2) = f_{xx}f_{yy}-(f_{xy})^2 = 1\times(-2+1)-1^2 = -2 < 0$, which implies that $(3/2,-1/2)$ is a saddle point.

**9.** Find the maximum and minimum values of $f(x,y) = 2x+3y$ subject to $g(x,y) = x^2+y^2-13 = 0$.

Using Lagrange condition $\nabla f = \lambda\nabla g$ we get $2 = 2\lambda x$ and $3 = 2\lambda y$. So $x = 1/\lambda$ and $y = 3/(2\lambda)$.
From $x^2+y^2-13 = 0$ we get $\frac{1}{\lambda^2}+\frac{9}{4\lambda^2}-13 = 0$, which leads to $\lambda^2 = 1/4$ and thus $\lambda = \pm 1/2$.
Critical points are given by $(x,y) = (1/\lambda,\ 3/(2\lambda))$, so there are two critical points $(2,3)$ and $(-2,-3)$.

Since the domain $g(x,y) = x^2+y^2-13 = 0$ is closed and bounded, $f$ achieve its global max at (2,3) with $f(2,3) = 13$ and achieve its global min at $(-2,-3)$ with $f(-2,-3) = -13$.

**10.** Evaluate $\displaystyle\int_0^1\int_1^2 \frac{1}{(x+2y)^3}\,dxdy.$

First, $\displaystyle\int_1^2 \frac{1}{(x+2y)^3}dx = \int_1^2 (x+2y)^{-3}dx = -\frac{1}{2}(x+2y)^{-2}\Big|_{x=1}^{x=2} = -\frac{1}{2}(2+2y)^{-2}+\frac{1}{2}(1+2y)^{-2}.$ Then we have

$$\int_0^1\int_1^2 \frac{1}{(x+2y)^3}dxdy = \int_0^1\left(-\frac{1}{2}(2+2y)^{-2}+\frac{1}{2}(1+2y)^{-2}\right)dy = \frac{1}{4}(2+2y)^{-1}-\frac{1}{4}(1+2y)^{-1}\Big|_0^1 = \frac{1}{4}\left(\frac{1}{4}-\frac{1}{3}-\frac{1}{2}+1\right) = \frac{5}{48}.$$