p1. Let C_1 and C_2 be convex sets in R^n with $C_1 \cap C_2 = \emptyset$. Prove that there exists $\boldsymbol{b} \neq \boldsymbol{0}$ with $\boldsymbol{b} \cdot \boldsymbol{u} \leq \boldsymbol{b} \cdot \boldsymbol{v}$, for all $\boldsymbol{u} \in C_1$ and $\boldsymbol{v} \in C_2$.

Proof. Let $C = \{\boldsymbol{u} - \boldsymbol{v} : \boldsymbol{u} \in C_1, \boldsymbol{v} \in C_2\}$. We first prove that C is convex. Suppose $\boldsymbol{x}, \boldsymbol{y} \in C$ and $\lambda \in [0, 1]$. By the definition of C, there exist $\boldsymbol{u}_x, \boldsymbol{u}_y \in C_1$ and $\boldsymbol{v}_x, \boldsymbol{v}_y \in C_2$ such that $\boldsymbol{x} = \boldsymbol{u}_x - \boldsymbol{v}_x$ and $\boldsymbol{y} = \boldsymbol{u}_y - \boldsymbol{v}_y$. Let $\boldsymbol{u} = \lambda \boldsymbol{u}_x + (1 - \lambda)\boldsymbol{u}_y$ and $\boldsymbol{v} = \lambda \boldsymbol{v}_x + (1 - \lambda)\boldsymbol{v}_y$. Since C_1, C_2 are convex, we must have $\boldsymbol{u} \in C_1$ and $\boldsymbol{v} \in C_2$ Therefore,

$$\begin{aligned} \lambda \boldsymbol{x} + (1 - \lambda)\boldsymbol{y} &= \lambda(\boldsymbol{u}_x - \boldsymbol{v}_x) + (1 - \lambda)(\boldsymbol{u}_y - \boldsymbol{v}_y) \\ &= (\lambda \boldsymbol{u}_x + (1 - \lambda)\boldsymbol{u}_y) - (\lambda \boldsymbol{v}_x + (1 - \lambda)\boldsymbol{v}_y) \\ &= \boldsymbol{u} - \boldsymbol{v} \in C, \end{aligned}$$

which proves that C is convex.

Notice that vector $\boldsymbol{0}$ is not in C, as $C_1 \cap C_2 = \emptyset$. In addition, by Theorem 5.1.7, $\overline{C}$ is convex. Next we show that there exists a nonzero vector $\boldsymbol{a}$ with $\boldsymbol{a}\boldsymbol{x} \leq \boldsymbol{a}\boldsymbol{0}$, for all $\boldsymbol{x} \in \overline{C}$. If $\boldsymbol{0} \notin \overline{C}$, this clearly follows from Theorem 5.1.5. If $\boldsymbol{0} \in \overline{C}$, by the lemma given in the hint, $\boldsymbol{0}$ is on the boundary of $\overline{C}$. Then the existence of $\boldsymbol{a}$ follows from Theorem 5.1.9 (as shown in class that we don't need to assume that $\overline{C}$ has an interior point).

Since $C \subseteq \overline{C}$, we have $\boldsymbol{a}\boldsymbol{x} \leq 0$, for all $\boldsymbol{x} \in C$. By the definition of C, it follows that $\boldsymbol{a}\boldsymbol{u} \leq \boldsymbol{a}\boldsymbol{v}$, for all $\boldsymbol{u} \in C_1$ and $\boldsymbol{v} \in C_2$. Thus we may take $\boldsymbol{b} = \boldsymbol{a}$, which satisfies the requirement.

5a. Notice that $f(x_1, x_2) = e^{-(x_1+x_2)}$, $g_1(x_1, x_2) = e^{x_1} + e^{x_2} - 20$ and $g_2(x_1, x_2) = -x_1$. It is straightforward to check that

$$Hf = e^{-(x_1+x_2)}I, \ \ Hg_1 = \begin{bmatrix} e^{x_1} & 0 \\ 0 & e^{x_2} \end{bmatrix}, \ \text{and } Hg_2 = \boldsymbol{0},$$

and that they are all positive semidefinate, for all $(x_1, x_2) \in R^2$. By Theorem 2.3.7, f, g_1, g_2 are convex functions, and thus the given program is convex. Moreover, $g_1(3, 0) < 0$ and $g_2(3, 0) < 0$, so the program is superconsistent. Finally, f, g_1 are exponential and g_2 is linear, so they have continuous first partial derivatives. Therefore, the program satisfies all conditions of (5.2.14).

The KKT condition is:

(1) $\lambda_1 \geq 0$,
(2) $\lambda_2 \geq 0$,
(3) $\lambda_1(e^{x_1} + e^{x_2} - 20) = 0$,
(4) $\lambda_2(-x_1) = 0$,
(5) $-e^{-(x_1+x_2)} + \lambda_1 e^{x_1} - \lambda_2 = 0$,
(6) $-e^{-(x_1+x_2)} + \lambda_1 e^{x_2} = 0$.

It follows from (6) that $\lambda_1 \neq 0$, so (3) can be simplified to

(3′) $e^{x_1} + e^{x_2} - 20 = 0$

If $x_1 = 0$, we deduce from (3′) that $e^{x_2} = 19$. Then we deduce from (6) that $\lambda_1 = \frac{1}{361}$. In this case, (5) implies $\lambda_2 = \frac{-19}{361}$, which does not satisfy (2). Therefore, $x_1 \neq 0$, and so, by (4), $\lambda_2 = 0$. Consequently, we deduce from (5-6) that $x_1 = x_2$. Now from (3′) we get $x_1 = x_2 = \ln 10$, and from (6) we get $\lambda_1 = \frac{1}{1000}$. In conclusion, (1-6) has a unique solution $x_1 = x_2 = \ln 10$, $\lambda_1 = \frac{1}{1000}$, $\lambda_2 = 0$. Since $x_1 = x_2 = \ln 10$ is feasible, (5.2.14) implies that it is the only solution to the given program.

5b. Just like in 5a, we can see that the given program satisfies the conditions of (5.2.14), so we can solve the program using the KKT condition:

(1) $\lambda_1 \geq 0$,
(2) $\lambda_2 \geq 0$,
(3) $\lambda_1(x_1^2 - x_2) = 0$,
(4) $\lambda_2(x_1 + x_2 - 2) = 0$,
(5) $2x_1 - 4 + 2\lambda_1 x_1 + \lambda_2 = 0$,
(6) $2x_2 - 4 - \lambda_1 + \lambda_2 = 0$.

We first observe that $\lambda_2 \neq 0$. This is because if $\lambda_2 = 0$, then (5-6) and (1) imply that $x_1 = 2/(1 + \lambda_1) > 0$ and $x_2 = 2 + \frac{1}{2}\lambda_1 \geq 2$. However, this is impossible since $g_2(x_1, x_2) = x_1 + x_2 \leq 2$. Thus $\lambda_2 \neq 0$ is proved. Consequently, (4) can be simplified as:

(4′) $x_2 = 2 - x_1$.

With this equation, (3) and (6) can be simplified as:

(3′) $\lambda_1(x_1 + 2)(x_1 - 1) = 0$,
(6′) $2x_1 + \lambda_1 - \lambda_2 = 0$.

If $x_1 = -2$, adding (5) and (6′) would give $\lambda_1 = -4$, which violates (1). So $x_1 \neq -2$, and thus (3′) implies that either $\lambda_1 = 0$ or $x_1 = 1$. In both cases, solving (4′), (5), and (6′) we end up with the same solution $x_1 = x_2 = 1$, $\lambda_1 = 0$, $\lambda_2 = 2$, which is the unique solution to (1-6). Since $x_1 = x_2 = 1$ is feasible, (5.2.14) implies that it is the only solution to the given program.