6a. To use KKT condition we express the problem as a convex program. In particular, $f = t_1^{-1}t_2^{-1}$, $g_1 = \frac{1}{2}t_1 + \frac{1}{2}t_2 - 1$, and $C = \{(t_1, t_2) : t_1 > 0, t_2 > 0\}$. It is clear that C is convex. It is also easy to check that $Hf = \frac{1}{t_1^3 t_2^3}\begin{bmatrix} 2t_2^2 & t_1 t_2 \\ t_1 t_2 & 2t_1^2 \end{bmatrix}$, $Hg_1 = \mathbf{0}$, and they are both positive semidefinite on C, which imply that both f and g_1 are convex on C. Therefore, the program is convex.

Observe that f and g_1 have continuous first partial derivatives on C. The program is also superconsistent since $(\frac{1}{2}, \frac{1}{2}) \in C$ and $g_1(\frac{1}{2}, \frac{1}{2}) < 0$. Therefore, all requirements in (5.2.14) are satisfied, so we may apply KKT condition.

(1) $\lambda_1 \geq 0$,
(2) $\lambda_1(\frac{1}{2}t_1 + \frac{1}{2}t_2 - 1) = 0$
(3) $-\frac{1}{t_1^2 t_2} + \frac{1}{2}\lambda_1 = 0$
(4) $-\frac{1}{t_1 t_2^2} + \frac{1}{2}\lambda_1 = 0$

Form (3-4) it is clear that $\lambda_1 \neq 0$ and $t_1 = t_2$. Thus we deduce from (2) that $t_1 = t_2 = 1$ is the optimal solution.

6b. To solve the problem as a geometric program we write it in the standard form. In particular, $g_0 = t_1^{-1}t_2^{-1}$ and $g_1 = \frac{1}{2}t_1 + \frac{1}{2}t_2$. We write the dual program:

$$\max\ (\frac{1}{\delta_1})^{\delta_1}(\frac{1}{2\delta_2})^{\delta_2}(\frac{1}{2\delta_3})^{\delta_3}(\delta_2 + \delta_3)^{\delta_2+\delta_3}$$

s.t. $\delta_1 = 1$, $-\delta_1 + \delta_2 = 0$, $-\delta_1 + \delta_3 = 0$, $\delta_1 > 0$, $\delta_2 \geq 0$, $\delta_3 \geq 0$.

Solving the equations we get $\delta_1 = \delta_2 = \delta_3 = 1$.

To get a solution to the original problem, we solve

$$f(\boldsymbol{t}) = v(\boldsymbol{\delta}) \quad \text{and} \quad \frac{u_1(\boldsymbol{t})}{\delta_2} = \frac{u_2(\boldsymbol{t})}{\delta_2}, \text{ (or } g_1(\boldsymbol{t}) = 1)$$

that is,

$$t_1^{-1}t_2^{-1} = 1 \quad \text{and} \quad \frac{t_1}{2} = \frac{t_2}{2},$$

which yields the optimal solution $t_1 = t_2 = 1$.

8b. Let $u_1 = x^{1/2}$, $u_2 = y^{-2}$, $u_3 = x^{-1}z$, $u_4 = x^{-1}w$, $u_5 = yz^{-1}$, and $u_6 = wz^{-1}$. Let $g_0 = x^{1/2} + y^{-2}$, $g_1 = x^{-1}z + x^{-1}w$, and $g_2 = yz^{-1} + wz^{-1}$. Then the dual program is

$$\max\ v(\boldsymbol{\delta}) = (\frac{1}{\delta_1})^{\delta_1}(\frac{1}{\delta_2})^{\delta_2}(\frac{1}{\delta_3})^{\delta_3}(\frac{1}{\delta_4})^{\delta_4}(\frac{1}{\delta_5})^{\delta_5}(\frac{1}{\delta_6})^{\delta_6}(\delta_3 + \delta_4)^{\delta_3+\delta_4}(\delta_5 + \delta_6)^{\delta_5+\delta_6}$$

s.t. $\delta_1 + \delta_2 = 1$
$\frac{1}{2}\delta_1 - \delta_3 - \delta_4 = 0$
$-2\delta_2 + \delta_5 = 0$
$\delta_3 - \delta_5 - \delta_6 = 0$
$\delta_4 + \delta_6 = 0$
$\delta_1 > 0, \delta_2 > 0$,
either $\delta_3 = \delta_4 = 0$ or both δ_3 and δ_4 are positive
either $\delta_5 = \delta_6 = 0$ or both δ_5 and δ_6 are positive

It follows from $\delta_4 + \delta_6 = 0$ that $\delta_4 = \delta_6 = 0$. Then the last two conditions imply $\delta_3 = \delta_5 = 0$. Now from $-2\delta_2 + \delta_5 = 0$ we get $\delta_2 = 0$, contradicting the requirement $\delta_2 > 0$. Therefore, the dual program has no feasible solution and thus, by Theorem 5.3.5, the original problem has no optimal solution. (When we apply 5.3.5 we need to verify that the original program is superconsistent. But this is clear since by taking $\boldsymbol{t} = (x, y, z, w) = (5, 1, 3, 1)$ we get $g_1(\boldsymbol{t}) < 1$ and $g_2(\boldsymbol{t}) < 1$.)

There is a more direct way to see that the given program does not have an optimal solution. Let $\boldsymbol{t} = (x, y, z, w)$ be any feasible solution. From $g_1(\boldsymbol{t}) \leq 1$ we get $x \geq z + w > z$; from $g_2(\boldsymbol{t}) \leq 1$ we get $z \geq y + w > y$. Combining these two we get $x > y$. Now it is straightforward to verify that

$$\bar{\boldsymbol{t}} = (\bar{x}, \bar{y}, \bar{z}, \bar{w}) = (\frac{2x+y}{3},\ \ y,\ \ \frac{x+2y}{3},\ \ \frac{x-y}{3})$$

is a feasible solution with $g_0(\bar{\boldsymbol{t}}) < g_0(\boldsymbol{t})$. What this says is that, for any feasible solution $\boldsymbol{t}$ there is always a better feasible solution $\bar{\boldsymbol{t}}$. Thus the program cannot have an optimal solution.

2a. Let $g(x_1, x_2) = x_1^2 - x_2 - 2$. Then

$$g^+ = \begin{cases} 0 & \text{if } x_1^2 \leq x_2 + 2 \\ x_1^2 - x_2 - 2 & \text{if } x_1^2 > x_2 + 2 \end{cases}$$

and $P_k = f + k \cdot (g^+)^2$. To minimize P_k, we compute

$$\nabla(g^+)^2 = 2g^+\nabla g^+ = \begin{cases} \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \text{if } x_1^2 \leq x_2 + 2 \\ (x_1^2 - x_2 - 2)\begin{bmatrix} 2x_1 \\ -1 \end{bmatrix} & \text{if } x_1^2 > x_2 + 2 \end{cases},$$

then

$$\nabla P_k = \begin{cases} \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \text{if } x_1^2 \leq x_2 + 2 \\ \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2k(x_1^2 - x_2 - 2)\begin{bmatrix} 2x_1 \\ -1 \end{bmatrix} & \text{if } x_1^2 > x_2 + 2 \end{cases}$$

and

$$Hf = \begin{cases} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \text{if } x_1^2 \leq x_2 + 2 \\ 2k\begin{bmatrix} 6x_1^2 - 2x_2 - 4 & -2x_1 \\ -2x_1 & 1 \end{bmatrix} & \text{if } x_1^2 > x_2 + 2. \end{cases}$$

First we find critical points by solving $\nabla P_k = \mathbf{0}$. Equivalently, we solve $\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2k(x_1^2 - x_2 - 2)\begin{bmatrix} 2x_1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ with $x_1^2 > x_2 + 2$. In other words, we need to solve

(1) $1 + 4k(x_1^2 - x_2 - 2)x_1 = 0$
(2) $1 - 2k(x_1^2 - x_2 - 2) = 0$
(3) $x_1^2 > x_2 + 2$.

From (2) we get $2k(x_1^2 - x_2 - 2) = 1$. By substituting this in (1) we get $1 + 2x_1 = 0$, which gives us $x_1 = -1/2$. Now from (2) we have $x_2 = -\frac{7}{4} - \frac{1}{2k}$. Since $\boldsymbol{x}^* = (-\frac{1}{2}, -\frac{7}{4} - \frac{1}{2k})$ satisfies (3), it is the only critical point.

To confirm that $\boldsymbol{x}^*$ is a minimizer, we consider Hf. By Theorem 1.2.9a, we only need to prove that $Hf(\boldsymbol{x})$ is positive semidefinate, for all $\boldsymbol{x}$. This is obvious if $x_1^2 \leq x_2 + 2$. In the case $x_1^2 > x_2 + 2$, we can check that all its principal minors are nonnegative:

$6x_1^2 - 2x_2 - 4 = 4x_1^2 + 2(x_1^2 - x_2 - 2) \geq 0$;
$1 \geq 0$; and
$(6x_1^2 - 2x_2 - 4) \cdot 1 - (-2x_1)^2 = 2(x_1^2 - x_2 - 2) \geq 0$.

By Theorem 1.3.4.d in the supplement, Hf is positive semidefinate and thus $\boldsymbol{x}^*$ is a global minimizer.

Finally, since $\lim_{k\to\infty} \boldsymbol{x}^* = (-\frac{1}{2}, -\frac{7}{4})$, we conclude that it is the optimal solution to the given problem.

Remark. Notice that f is not coercive and it does not satisfy the lower bound condition in 6.2.3. However, the penalty method still produces the optimal solution.