Supplement

Chapter 1.

The next result says that, as long as the domain is closed and the function is continuous, a global maximizer and a global minimizer must exist.

(1.1.7) Theorem. *Suppose that $f(x)$ is continuous on a closed interval. Then $f(x)$ has a global maximizer and a global minimizer.*

Proof. Hard. ∎

The following result tells how to find a global maximizer and a global minimizer if the domain is closed.

(1.1.8) Theorem. *Suppose that $f(x)$ is differentiable on a finite interval $[a, b]$. If x_1, x_2, ..., x_t are the critical points, then both the global maximizer and global minimizer belong to $\{a, b, x_1, x_2, ..., x_t\}$.*

Proof. The result follows obviously from Theorem 1.1.4 and Theorem 1.1.7. ∎

If the domain is not closed, the next result tells how to find a global maximizer and a global minimizer.

(1.1.9) Theorem. *Suppose that $f(x)$ is differentiable on $(-\infty, \infty)$. If there are two points x_1 and x_2 such that both $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$ are between $f(x_1)$ and $f(x_2)$, then some local maximizer is a global maximizer and some local minimizer is a global minimizer.*

Proof. The assumption on limit implies that there exists a closed interval $[a, b]$ such that, for all x outside $[a, b]$, $f(x)$ is between $f(x_1)$ and $f(x_2)$. It follows that the global maximizer and global minimizer belong to $[a, b]$. The the result follows from Theorem 1.1.8. ∎

The next is a theorem from linear algebra. Let A be a square matrix. If B is obtained from A by deleting rows and columns of the same indices, then the determinant of B is called a *principal minor* of A.

(1.3.4.d) Theorem. *A symmetric matrix is positive semidefinite if and only if all its principal minors are nonnegative.*

As a corollary, the following characterizes indefinite 2×2 symmetric matrices.

(1.3.4.e) Corollary. *Symmetric matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is indefinite if and only if $a_{11}a_{22} < 0$ or $\det(A) < 0$.*

Proof. It follows from the last Theorem that A is positive semidefinite if and only if $a_{11} \geq 0$, $a_{22} \geq 0$, and $\det(A) \geq 0$. Similarly, A is negative semidefinite (that is, $-A$ is positive semidefinite) if and only if $-a_{11} \geq 0$, $-a_{22} \geq 0$, and $\det(-A) = \det(A) \geq 0$. Therefore, A is indefinite (that is, A is neither positive semidefinite nor negative semidefinite) if and only if a_{11}, a_{22} have different signs or $\det(A) < 0$. ∎

The next result gives a method to prove that a point is not a local maximizer/minimizer.

(1.3.9) Theorem. *If $\mathbf{x}^* = (x_1^*, x_2^*, ..., x_n^*)$ is a local minimizer of $f(x_1, x_2, ..., x_n)$, then x_1^* is a local minimizer of $f(x_1, x_2^*, ..., x_n^*)$. Equivalently, if x_1^* is not a local minimizer of $f(x_1, x_2^*, ..., x_n^*)$, then $\mathbf{x}^*$ is not a local minimizer of $f(\mathbf{x})$.*

Proof. Obvious. ∎

The next is a simple observation that can be used to show that a function has neither global maximizer nor global minimizer.

(1.3.10) Theorem. *Let $a, x_2^*, ..., x_n^*$ be fixed numbers, where a could be ∞ or $-\infty$. If $\lim_{x_1\to a} f(x_1, x_2^*, ..., x_n^*) = \infty$, then $f(\mathbf{x})$ has no global maximizer. Similarly, if we have $\lim_{x_1\to a} f(x_1, x_2^*, ..., x_n^*) = -\infty$, then $f(\mathbf{x})$ has no global minimizer.*

Proof. If f has a global maximizer $\mathbf{x}^*$, then $f(\mathbf{x}) \leq f(\mathbf{x}^*)$ holds for all $\mathbf{x}$. Hence, $\lim_{x_1\to a} f(x_1, x_2^*, ..., x_n^*) = \infty$ will never happen. Therefore, $\lim_{x_1\to a} f(x_1, x_2^*, ..., x_n^*) = \infty$ implies that f has no global maximizer. The proof for global minimizer is similar. ∎