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Algebraic Geometry

Lecture 1.

ALGEBRAIC VARIETIES.

1. Notation

A field with which we are working will be denoted by k. It is assumed that it is algebraically closed: $k = \overline{k}$, and it can be $\mathbb{C}$, $\overline{\mathbb{F}_p}$, or $\overline{\mathbb{Q}}$. We'll work with polynomial rings $k[x_1, \ldots, x_n]$ and its elements $f \in k[x_1, \ldots, x_n]$, such that $f = \sum_\alpha c_\alpha x^\alpha$, where $c_\alpha \in k$, $x^\alpha = x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$. All rings will be unitary and commutative.

Definition 1.1. An affine n-space $\mathbb{A}^n_k = \mathbb{A}^n(k) = k^n$.

Definition 1.2. An affine algebraic set $X \subseteq k^n$ is a subset of affine space, such that for some subset of polynomials $\{f_\mu\}_{\mu \in M}$ we have $f_\mu(a) = 0$ for any $a \in X$ and any $\mu \in M$.

Examples of affine algebraic sets are lines, hyperplanes and quadrics. When there is a given set of polynomials, then the affine algebraic set associated with it will be denoted as

$$V(\{f_\mu\}) = \{a \in k^n : f_\mu(a) = 0, \forall \mu \in M\}.$$

As we know, a subset of polynomials generates an ideal in the ring $k[x_1, \ldots, x_n]$. There is a simple result about affine algebraic sets of subsets and ideals:

Lemma 1.3.

$$V(\{f_\mu\}) = V(\langle f_\mu \rangle).$$

Proof. As all the polynomials from $\{f_\mu\}$ belong to the $\langle f_\mu \rangle$, then all common zeroes of the ideal should be in the common zero set of the polynomials: $V(\langle f_\mu \rangle) \subseteq V(\{f_\mu\})$. To show the reverse inclusion, take a point $a \in V(\{f_\mu\})$: we know that $f_\mu(a) = 0$ for all $\mu \in M$. Then any polynomial g from the ideal is a combination of the polynomials from the set: $g = \sum_\mu h_\mu(x) f_\mu(x)$, where each $h_\mu \in k[x_1, \ldots, x_n]$. So $g(a) = \sum_\mu h_\mu(a) f_\mu(a) = 0$, and $V(\{f_\mu\}) \subseteq V(\langle f_\mu \rangle)$ $\square$

Therefore we have a correlation between ideals and affine algebraic sets.

Definition 1.4. If $I \subseteq k[x_1, \ldots x_n]$ is an ideal, then the radical of I is

$$\sqrt{I} = \{f \in k[x_1, \ldots x_n] : f^m \in I, \text{ for some interger } m\}$$

It is easy to show that the radical of the ideal is an ideal as well.

Definition 1.5. If $I = \sqrt{I}$, then we say I is a radical ideal.

Example 1.6. Take $R = \mathbb{Z}$. The ideal here are in the form $I = \langle n \rangle = \langle p_1^{r_1} \ldots p_s^{r_s} \rangle$. Its radical is $\sqrt{I} = \langle p_1 \ldots p_s \rangle$

Lemma 1.7. $V(I) = V(\sqrt{I})$.

Proof. $I \subseteq \sqrt{I}$, thus $V(\sqrt{I}) \subseteq V(I)$. Choose arbitrary $a \in V(I)$ and $g \in \sqrt{I}$, so $g^m \in I$ for some integer m and $(g(a))^m = 0$. But it could be 0 only if $g(a) = 0$, hence $a \in V(\sqrt{I})$. □

Theorem 1.8. *(Hilbert basis theorem) Every ideal of $k[x_1, \ldots, x_n]$ is finitely generated: $I = \langle f_1, \cdots, f_n \rangle$. Therefore, every affine algebraic set can be defined by a finite number of polynomial equations.*

Proposition 1.9. *If R is a commutative ring and P is its proper ideal, then the following statements are equivalent:*

1. *R/P is an intergral domain.*
2. *Ideal P is prime, meaning that if $x, y \in R$ and $xy \in P$, then $x \in P$ or $y \in P$.*

Proposition 1.10. *If R is a commutative ring and M is an ideal, then the following statements are equivalent:*

1. *R/M is a field.*
2. *Ideal M maximal, so if $M \subseteq I \subseteq R$, where I is another ideal, then $I = M$ or $I = R$.*

Definition 1.11. The set of all prime ideals of R is denoted as Spec(R). The set of all maximal ideals of the ring is denoted as Max(R).

Note that Max(R) $\subseteq$ Spec(R).

Example 1.12. $R = k$, a field. Then Max(R)=Spec(R)=$\{\langle 0 \rangle\}$, as $k/\langle 0 \rangle = k$, the field. Note that $\langle 0 \rangle \in$ Spec(R) if and only if R is an intergral domain, since then $R/\langle 0 \rangle$ is an integral domain.

Example 1.13. $R = \mathbb{Z}$, Spec($\mathbb{Z}$) $= \{\langle 0 \rangle, \langle 2 \rangle, \langle 3 \rangle, \ldots, \langle p \rangle, \ldots\}$, where p is a prime number. Checking: $\mathbb{Z}/\langle p \rangle = \mathbb{F}_p$, which is a field.

Example 1.14. $R = k[x]$, recall that it is a PID. Spec($k[x]$)=$\{\langle 0 \rangle, \ldots, \langle f \rangle, \}$, where f is an irreducible polynomial, $k[x]/\langle f \rangle$ is an integral domain, even if k is not algebraically closed.

Example 1.15. $R = k[x, y]$, Assume that k is algebraically closed.. Spec($k[x]$)=$\{\langle 0 \rangle, \ldots, \langle f \rangle, \} \cup$ Max(R) where f is an irreducible polynomial and Max(R) $= \{\langle x - \alpha, y - \beta \rangle\}$, $(\alpha, \beta) \in k^2$.