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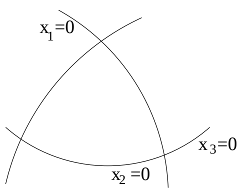
Algebraic Geometry

Lecture 14.

PROJECTIVE CLOSURE.

Consider an open set in the Zariski topology

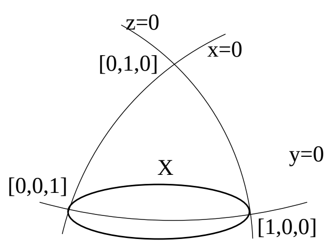
$$U_i = \mathbb{P}^n \setminus V(x_i) = \{[x_0, \dots, x_n] \in \mathbb{P}^n | x_i \neq 0\}.$$



Note that a conventional picture for a projective space (without a common zero for all variables) is on the left, and the algebraic set $V(x_j) \cong \mathbb{P}^{n-1}$. Consider a canonical isomorphism $\phi_i : U_i \longrightarrow \mathbb{A}_n$, by mapping $[x_0, x_1, \dots, x_n] \rightarrow (\frac{x_0}{x_i}, \frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i})$. We'll see today that ϕ_i is a homeomorphism for Zariski topology.

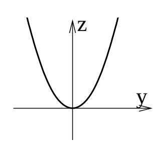
Given a projective algebraic set $X \subseteq \mathbb{P}^n$, we can "view it" in the ith coordinate chart $\phi_i(U_i \cap X) = \mathbb{A}^n$.

Example 0.1. $X = V(y^2 - xz) \subset \mathbb{P}^2$.

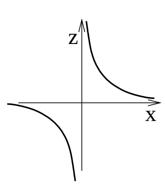


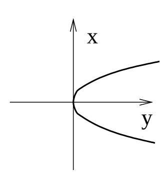
Now we can consider it in the charts
$U_0 = \mathbb{P}^2 \setminus V(x) = \{[x, y, z] \in \mathbb{P}^2 | x \neq 0\}$,
$U_1 = \mathbb{P}^2 \setminus V(y) = \{[x, y, z] \in \mathbb{P}^3 | y \neq 0\}$, and
$U_3 = \mathbb{P}^2 \setminus V(z) = \{[x, y, z] \in \mathbb{P}^2 | z \neq 0\}$.

In U_0 we have $\phi_0(X \cap U_0) = V(y^2 - z) \subset \mathbb{A}^2$.
(Remark: when you have a homogeneous polynomial, it is enough to take $x = 1$ for the image of it under ϕ_0.)

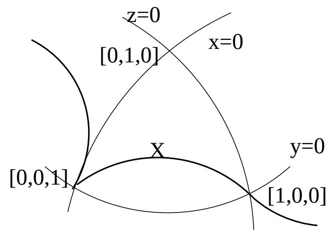


In U_1 we have $\phi_1(X \cap U_1) = V((1 - zx) \subset \mathbb{A}^2$, and in U_2 we have $\phi_2(X \cap U_2) = V(y^2 - x) \subset \mathbb{A}^2$. (See the corresponding pictures below.)

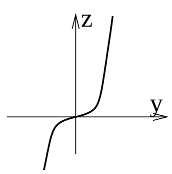


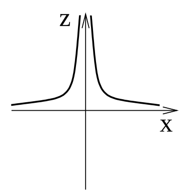


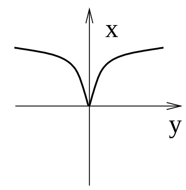
Example 0.2. $X = V(y^3 - x^2 z) \subset \mathbb{P}^2$.



In U_0 we have $\phi_0(X \cap U_0) = V(y^3 - z) \subset \mathbb{A}^2$,
in U_1 we have $\phi_1(X \cap U_1) = V(x^2 z - 1)$,
and in U_2 we have $\phi_2(X \cap U_2) = V(y^3 - x^2)$.







Let X be a projective algebraic set in $\mathbb{P}^n$.

Lemma 0.3. *If* $X = V(F_1, F_2, \ldots, F_s) \subseteq \mathbb{P}^n$, *and* $\phi_i(X \cap U_i) = V(f_1, f_2, \ldots, f_s) \subseteq \mathbb{A}^n$, *where* $f_j(x_0, \ldots, \widehat{x_i}, \ldots, x_n) = F_j(x_0, \ldots, x_i = 1, \ldots, x_n)$, *then*

$$X = (X \cap U_i) \cup (X \cap V(x_i)),$$

Remark: The first set in the union above is an affine piece in the ith chart, the second is a piece at infinity, in a hyperplane at $x_i = 0$, which homeomorphic to $\mathbb{P}^{n-1}$.

Let's fix a chart for $\boxed{i = 0}$, and consider an inverse of ϕ_0. We can see that $\phi_0^{-1} : \mathbb{A}^n \xrightarrow{\sim} U_0 \hookrightarrow \mathbb{P}^n$.

Definition 0.4.
(1) Let $X \subseteq \mathbb{A}^n$ be affine algebraic set. The closure of $\phi_0^{-1}(X)$ in $\mathbb{P}^n$ is called the projective closure of X, (relative to a chart) and denoted by $\overline{X}$. $X_\infty = \overline{X} \cap H_\infty$, $H_\infty = V(x_0) \cong \mathbb{P}^{n-1} \subset \mathbb{P}^n$.
(2) Let J be an ideal of $k[x_1, \ldots, x_n]$. Define $J^h = \langle f^h | f \in J \rangle$, where

$$f^h(x_0, x_1, \ldots, x_n) := x_0^{\deg f} f\left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \ldots, \frac{x_n}{x_0}\right).$$

(3) $J_\infty = \langle g_\infty \mid g \in J \rangle$, where $g_\infty(x_1, \ldots x_n) = g(x_0 = 0, x_1, \ldots x_n)$.
(4) If $X \subset \mathbb{P}^n$ is a projective algebraic set, denote $X^a = \phi_0(X \cap U_0) \subseteq \mathbb{A}^n$ - an affine part of X.
(5) If J is a homogeneous ideal of $k[x_0, x_1, \ldots, x_n]$, define an ideal J^a of $k[x_1, \ldots, x_n]$ as $J^a = \langle F^a | \ F \in J, F \text{ is homogeneous } \rangle$, where $F^a = F(x_0 = 1, x_1, \ldots, x_n)$.

Lemma 0.5. *If* $F \in k[x_0, x_1, \ldots, x_n]$ *is a homogeneous polynomial,* $x_0 \nmid F$, *then* $F = (F^a)^h$.

Consider an example when the lemma does not hold: $F = x^2y^3 + x^3yz + x^2z^3$, $F^a = y^3 + yz + z^3$, $(F^a)^h = y^3 + xyz + z^3$.

Proposition 0.6.
(1) *If* J *is an ideal of* $k[x_1, \ldots, x_n]$, *and* $X = V(J) \subseteq \mathbb{A}^n$, *then* $I(\overline{X}) = I(X)^h$, $V(J^h) = \overline{X}$, $I(X_\infty) = \sqrt{I(X)_\infty}$, $V(J_\infty) = X_\infty$.

(2) *If J is a homogeneous ideal of $k[x_0, x_1, \dots, x_n]$, $X = V(J) \subseteq \mathbb{P}^n$, then $I(X^a) = I(X)^a$, $V(J^a) = X^a$.*

Warning: If $J = \langle f_1, \dots, f_s \rangle \subseteq k[x_1, \dots, x_n]$, (can be not homogeneous), it is not true in general that $J^h = \langle f_1^h, \dots, f_s^h \rangle$.

Example 0.7. Let $X = \{(t, t^2, t^3) |\ t \in k\} \subseteq \mathbb{A}^3$. Note that $X = V(y - x^2, z - xy)$. Consider variables $[w, x, y, z]$ in $\mathbb{P}^3$. Then

$$f_1^h = wy - x^2, \quad f_2^h = zw - xy, \quad V(f_1^h, f_2^h) = \overline{X} \cup \{\text{line } w = x = 0\},$$

and if we denote $J = \langle y - x^2, z - xy \rangle$, then

$$J^h = \langle wy - x^2, zw - xy, y^2 - xz \rangle \text{ and } \overline{X} = V(wy - x^2, zw - xy, y^2 - xz).$$