Lecture 17: Morphisms II

17.1 A Basis for the Zariski Topology

We begin with a proposition:

Proposition 1 *Let X be an irreducible Zariski-closed subset of $\mathcal{A}^n$, and f an element of its coordinate ring $k[X]$. Let X_f be the complement of the zero locus of f in X; that is, $X_f = \{x \in X : f(x) \neq 0\}$. Then X_f is an affine open set in the Zariski topology on X and $k[X_f] \cong k[X]_f$.*

First some reminders and remarks before proving the proposition. The Zariski topology on X is the relative topology inherited from $\mathcal{A}^n$. By affine, we mean that X_f is isomorphic as a quasiprojective algebraic set to a Zariski-closed subset of some affine space (say $\mathcal{A}^m$). Also recall that $k[X]_f$ is the localization of $k[X]$ with respect to the multiplicatively closed set $S_f = \{\frac{1}{f^j} : j \geq 0\}$.

Now for the proof. It is easy enough to see that X_f is open in X, since it is the complement of the closed set $V(f)$, the zero-locus of f. More precisely, if we choose generators $g_1, ..., g_r$ of $I(X)$ (so that $X = V(g_1, ..., g_r) \subset \mathcal{A}^n$), and a representative F of f in $k[x_1, ..., x_n]$, then as a subset of $\mathcal{A}^n$,

$$X_f = \{x \in \mathcal{A}^n : g_i(x) = 0 \forall i, f(x) \neq 0\} = V(g_1, ..., g_r) \backslash V(f) \tag{1}$$

Thus X_f is the intersection of $V(g_1, ..., g_r) = X$ and the Zariski-open set $\mathcal{A}^n \backslash V(f)$, so X_f is open in X by the definition of the relative topology.

It is not so obvious that X_f is affine. To prove this, we exhibit an isomorphism between X_f and a Zariski-closed subset of $\mathcal{A}^{n+1}$. Let $\mathcal{A}^{n+1} = \mathcal{A}^n \times \mathcal{A}^1$ have coordinates $(x_1, ..., x_n, t)$. Define an algebraic set $Y \subset \mathcal{A}^{n+1}$ by

$$Y = V(g_1(x), ..., g_r(x), tf(x) - 1) \tag{2}$$

Note that $g_1, ..., g_r$ are the same polynomials that define X as a variety in $\mathcal{A}^n$. We think of $g_1, ..., g_r$ as elements of $k[x_1, ..., x_n, t]$ in which the new variable t does not appear. The last polynomial $tf(x) - 1$ is the only one which involves t.

The restriction to Y of the projection map $\pi : \mathcal{A}^{n+1} \longrightarrow \mathcal{A}^n$ onto the first n coordinates of $\mathcal{A}^n$ (sending $(x_1, ..., x_n, t)$ to $(x_1, ..., x_n)$) is obviously a morphism of quasiprojective varieties (it is given by polynomials). The inverse function $\psi : X_f \longrightarrow Y$ is defined by

$$\psi(x_1, ..., x_n) = (x_1, ..., x_n, \frac{1}{f(x_1, ..., x_n)}) \tag{3}$$

We check that $\pi \circ \psi = Id_{X_f}$ and $\psi \circ \pi = Id_Y$. One direction is obvious. Let $x \in X_f$. Under the composition $\pi \circ \psi$:

$$x \mapsto (x, \frac{1}{f(x)}) \mapsto x \tag{4}$$

Let $(y,t) \in Y$. Under the reverse composition $\psi \circ \pi$:

$$(y,t) \mapsto y \mapsto (y, \frac{1}{f(y)}) \tag{5}$$

Since $Y = V(g_1, ..., g_r, tf - 1)$, we have $tf(y) - 1 = 0 \forall y \in Y$, so $t = \frac{1}{f}$ on Y. Thus, $\psi \circ \pi = Id_Y$. This also shows that the inverse map ψ is a morphism, since it is given by polynomials on Y.

We have proved that X_f is affine. It remains to show that $k[X_f] \cong k[X]_f$. For any commutative ring A with identity and any element $f \in A$, the localization $A_f \equiv S_f^{-1}A$ (where S_f is the multiplicatively closed set $\{\frac{1}{f^j} : j \geq 0\}$) is isomorphic to $\frac{A[t]}{<tf-1>}$. Now by definition, $Y = V(g_1(x), ..., g_r(x), tf(x) - 1)$, so by the Nullstellensatz, $I(Y) = \sqrt{< g_1, ..., g_r, tf - 1 >}$, so if we can show that $< g_1, ..., g_r, tf - 1 >$ is a radical ideal, then

$$k[X_f] \cong k[Y] = \frac{k[x_1, ..., x_n, t]}{I(Y)} = \frac{k[x_1, ..., x_n, t]}{< g_1, ..., g_r, tf - 1 >} = \frac{k[X][t]}{< tf - 1 >} = k[X]_f \tag{6}$$

Note that $< g_1, ..., g_r, tf - 1 >$ is radical iff the quotient ring $\frac{k[x_1,...,x_n,t]}{<g_1,...,g_r,tf-1>}$ has no nilpotent elements. We have the following lemma:

Lemma 2 *Suppose A is an integral domain and $S^{-1}A$ a localization. Then $S^{-1}A$ has no nilpotent elements.*

Proof: Suppose $\frac{a}{s} \in S^{-1}A$ is nilpotent ($a \neq 0$ by assumption). Then for some n, $(\frac{a}{s})^n = \frac{a^n}{s^n} = 0$. Recall that $\frac{a}{s}$ is by definition the equivalence class of (a, s) under the equivalence relation

$$(x, y) \sim (w, z) \leftrightarrow \exists u \in S, u \neq 0, \ni u(zx - yw) = 0 \tag{7}$$

Thus, $\frac{a^n}{s^n} = 0$ means $(a^n, s^n) \sim (0, 1)$, which means $\exists u \in S \ni ua^n = 0$, contradicting the fact that A is an integral domain. Thus $S^{-1}A$ has no nilpotent elements.

□

Now $\frac{k[x_1,...,x_n,t]}{<g_1,...,g_r,tf-1>}$ is the localization $k[X]_f$ of $k[X]$, which is an integral domain since X is a variety (irreducible). Thus $\frac{k[x_1,...,x_n,t]}{<g_1,...,g_r,tf-1>}$ has no nilpotent elements, and $< g_1, ..., g_r, tf - 1 >$ is radical. This completes the proof.

Proposition 3 *Let X be an irreducible Zariski-closed subset of $\mathcal{A}^n$. The sets $\{X_f : f \in k[X]\}$ defined above form a basis of open sets for the Zariski topology on X.*

Proof: Let W be a Zariski-open subset of X. Then $W = X \backslash V(J)$ where $V(J)$ is the algebraic set corresponding to some radical ideal $J \subset k[x_1, ..., x_n]$. By DeMorgan's laws:

$$W = X \backslash V(J) = X \backslash \bigcap_{f \in J} V(f) = \bigcup_{f \in J} (X \backslash V(f)) = \bigcup_{f \in J} X_f \tag{8}$$

Thus for any open set $W \subset X$ and any point $x \in W$, there is an affine open set X_f such that $x \in X_f \subset W$. So the sets X_f form a basis for the Zariski topology on X.

□

Corollary 4 *Every quasiprojective algebraic set is a finite union of affine algebraic varieties.*

Proof: Given a quasiprojective algebraic set $Y \subset \mathcal{P}^n$, consider the intersections of Y with the $n+1$ coordinate chart domains U_i (where $U_i = \{[x_0, ..., x_n] \in \mathcal{P}^n : x_i \neq 0\} = \mathcal{P}^n \backslash V(x_i)$. Each intersection $Y \bigcap U_i$ is (isomorphic to) a Zariski-open subset $W_i = X_i \backslash V(J)$ of some Zariski-closed set $X_i \in \mathcal{A}^n$. For a particular X_i and W_i (call them X and W), we have $W = \bigcup_{f \in J}(X \backslash V(f)) = \bigcup_{f \in J} X_f$ by the proof of Proposition 3, and there is a finite subcover of X_f's (which are affine) since the ideal J is finitely generated. Since the inverse images (under the chart maps) of the $n+1$ sets W_i cover X and since each W_i has a finite cover by affine algebraic varieties, so does Y.

□

17.2 The Local Ring Revisited

Definition 5 *Let X be an affine algebraic set and $p \in X$ a point. Then $\mathcal{O}_{X,p}$ (the local ring on X of functions regular at p), is defined by: $\mathcal{O}_{X,p} = k[X]_{M_p}$, where M_p is the maximal ideal corresponding to p via the Nullstellensatz.*

(Recall that we can localize any ring R in any prime ideal P: $R_P = S_P^{-1}R$, where S_P is the multiplicatively closed set $R \backslash P$.)

We already have a definition (15.8) for the local ring on X of functions regular at p for X a *quasiprojective variety*: $\mathcal{O}_{X,p} = \{f \in k(\overline{X}) : f \text{ regular at } p\} = \{\frac{G}{H} : G, H \in k[\overline{X}], G(p) \neq 0\}$. Here $\overline{X}$ is any projective closure of X and G and H are homogeneous polynomials of the same degree. If X is an affine algebraic variety, we may express the regular functions as ratios of functions in the coordinate ring $k[X]$ without resorting to the projective closure. Definition 5 above deals with *affine algebraic sets.* Thus our new definition is less general than the original one in the sense that X is assumed to be affine, but more general in the sense that X is no longer assumed to be irreducible. If X is both irreducible and affine (an affine algebraic variety), we must check that the two definitions coincide. This fact is the first part of the following proposition:

Proposition 6 *Definition 4 agrees with the definition of $\mathcal{O}_{X,p}$ given in Lecture 15 in the case that X is an affine algebraic variety. Moreover, for X an affine algebraic set, $\mathcal{O}_{X,p}$ only depends on the neighborhood of p. That is, if U is an affine open subset of X containing p, then $\mathcal{O}_{U,p} = \mathcal{O}_{X,p}$.*

For an affine algebraic variety, $\mathcal{O}_{X,p}$ is the set of rational functions $\frac{f}{g}$, where f and g are elements of $k[X]$ which do not vanish at p. M_p is the maximal ideal of functions in the coordinate ring $k[X]$ which *do* vanish at p. Thus $k[X]_{M_p} = S_{M_p}^{-1}k[X]$ where

$S_{M_p} = k[X]\backslash M_p$. Thus $k[X]_{M_p} = \{\frac{f}{g} : f, g \in k[X], g(p) \neq 0\} = \mathcal{O}_{X,p}$ according to the original definition.

In order to show that $\mathcal{O}_{X,p}$ depends only on the neighborhood of x, we need the following lemma:

Lemma 7 *Let A be a ring, $S \subset A$ a multiplicatively closed set, $P \in Spec(A)$ a prime ideal, and suppose $P \bigcap S$ is empty. Then*

$$A_P \cong \overline{A}_{\overline{P}} \tag{9}$$

where $\overline{A} \equiv S^{-1}A$ and $\overline{P} \equiv PS^{-1}A$.

We proved something similar to this is Lecture 9. The proof is not difficult, but is omitted here.

Back to showing that $\mathcal{O}_{X,p}$ depends only on the neighborhood of x. It is sufficient to show this for a basic open set. Recall that basic open sets are sets of the form $U = X_f = X\backslash V(f)$ for some $f \in k[X]$. Using the language of Lemma 7, we set $A = k[X]$, $S = \{f^j : j \geq 0\}$, $P = M_p$. Then $A_P = k[X]_{M_p} = \mathcal{O}_{X,p}$ and $\overline{A}_{\overline{P}} = k[X_f]_{\overline{M_p}} = \mathcal{O}_{U,p}$. By Lemma 7, $\mathcal{O}_{X,p} = A_P \cong \overline{A}_{\overline{P}} = \mathcal{O}_{U,p}$. This completes the proof.

17.3 Summary and Further Results

We now gather together and extend some of what we have proved:

Scholium 8 *1. Every quasiprojective algebraic set X is a finite union of affine open sets.*

2. The local ring $\mathcal{O}_{X,p}$ depends only on affine neighborhoods of p in X.

3. Let $f : X \longrightarrow Y$ be a morphism of quasiprojective varieties. Then we can find coverings $X = \bigcup_\alpha X_\alpha$, $Y = \bigcup_\beta Y_\beta$ where X_α, Y_β are affine and open such that $f|_{X_\alpha} : X_\alpha \longrightarrow Y_\beta$ is a morphism of affine algebraic varieties with induced k-algebra homomorphism $f^* : k[Y_\beta] \longrightarrow k[X_\alpha]$. (Here β depends on α and is not unique in general).

4. Thus, f induces a homomorphism of local rings (which we also label f^*):

$$f^* : \mathcal{O}_{Y,f(p)} \longrightarrow \mathcal{O}_{X,p} \tag{10}$$

for all $p \in X$, such that $f^*(M_{Y,f(p)}) \subset M_{X,p}$ (where $M_{Y,f(p)}$ is the maximal ideal on Y at $f(p)$ and $M_{X,p}$ is the maximal ideal on X at p). In particular, if f is an isomorphism, then $\mathcal{O}_{Y,f(p)} = \mathcal{O}_{X,p}$ for all $p \in X$.

In (4) above, the inclusion $f^*(M_{Y,f(p)}) \subset M_{X,p}$ comes from the simple fact that if a function $G : Y \longrightarrow k$ vanishes at $f(p)$ in Y, then the pullback $G \circ f$ vanishes at p. This also implies that for a morphism $f : X \longrightarrow Y$ and an open set $U \subset Y$, $\mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(f^{-1}(U))$ under f^*.

We conclude with a theorem and some consequences. The proof is not included here, but can be found in Hartshorne Chapter I.

Theorem 9 *If X is an affine algebraic variety, then $\mathcal{O}_X(X) = k[X]$. If X is a quasiprojective variety, then $\mathcal{O}_X(X)$ includes only constant functions.*

In particular, this shows that no projective variety is affine, unless it is a single point. A manifestation of the second part of the theorem is that there are no nonconstant holomorphic functions on the Riemann sphere (or any other compact complex manifold).