LECTURE 15

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1. Morphisms I

Definition 1.1. A scheme X is noetherian if it is a finite union of open subsets $X \cong \text{Spec} R_i$, where each R_i is a noetherian Ring.

Definition 1.2 (Subscheme)**.** Let $(X, \mathcal{O}_X)$ be a scheme.

1. An open subscheme is a pair of the form $(U, \mathcal{O}_X|U)$ where $U \subset X$ is an open subset.
2. A closed subscheme is a pair $(Y, \mathcal{O}_Y)$ where $Y \subset X$ is a closed subset, and $\mathcal{O}_Y = \mathcal{O}_X/\mathcal{J}$ with $\mathcal{J}$ a quasi-coherent sheaf of ideals.
3. A locally closed subscheme is a closed subscheme of an open subscheme.

Recall: $\mathcal{M}|U = \tilde{\mathcal{M}}$ where M is an R-module. $\tilde{\mathcal{M}}(U_f) := \mathcal{M}_f$ where $U_f = \{x \in U : f(x) \neq 0 \text{ in } \kappa(X)\}$
Also $\tilde{R} = \mathcal{O}_U$ and $\tilde{\mathcal{M}}_{[p]} = \mathcal{M}_p$.
The following are detail of part 2 of the last definition.
$\mathcal{J}|U =$ Spec $R = \tilde{J}$ for $J \subseteq R$ an ideal. There exists a 1-1 correspondence between the closed subscemes of Spec R and the ideals $J \subseteq R$.
Observe the following: $V(J) = \{[p] | p \supset J\} \stackrel{1-1}{\longleftrightarrow}$ Spec (R/J) by the Lattice isomorphism theorem. Notice $V(J) \cong$ Spec (R/J) is a homeomorphism. Therefore $V(J)$ carries the structure of a scheme $(Y, \mathcal{O}_Y)$. Namely $Y =$ Spec (R/J), and $\mathcal{O}_Y = \mathcal{O}_{\text{Spec }(R/J)}$.

Definition 1.3. Let $\Psi : X \to Y$ be a continuous map of topological spaces. Let $\mathfrak{f}$ be a sheaf on X. We can define a sheaf $\Psi_*\mathfrak{f}$ on Y as follows:
$\forall U \subseteq Y$ open, $\Psi_*\mathfrak{f}(U) := \mathfrak{f}(\Psi^{-1}U)$. This is called the direct image.

Definition 1.4. A morphism of locally ringed spaces is a pair $(\Psi, \Psi^\sharp)$ where

1. $\Psi : X \to Y$ is a continuous map.
2. $\Psi^\sharp : \mathcal{O}_Y \to \Psi_*\mathcal{O}_X$ where the following axiom holds.

Axiom 1.5. For any $x \in X$, the canonical map:
$\Psi^\sharp_x : \mathcal{O}_{Y,\Psi(x)} \xrightarrow{\Psi^\sharp} (\Psi_*\mathcal{O}_X)_{\Psi(x)} \xrightarrow{\text{canonical}} \mathcal{O}_{X,x}$
and
$\Psi^\sharp_x(\mathcal{M}_{Y,\Psi(x)}) \subseteq \mathcal{M}_{X,x}$

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Example 1.6. Let X, Y be a C^∞ manifolds. Let $\Psi : X \to Y$ be a C^∞ map(morphism) that is continuous. Then
$\mathcal{O}_X$ = sheaf of $C^\infty(\mathbf{R})$ functions on X
$\mathcal{O}_Y$ = sheaf of $C^\infty(\mathbf{R})$ functions on Y
We then have the map $\Psi^\sharp : \mathcal{O}_Y \to \Psi_X \mathcal{O}_X$, the pullback of C^∞ functions.

$$\begin{array}{ccccc} \Psi^{-1}U & \longrightarrow & U & \xrightarrow{f} & \mathbf{R} \\ \cap\big\downarrow & & \cap\big\downarrow & & \\ X & \xrightarrow{\Psi} & Y & & \end{array}$$

Also

$$\begin{array}{ccccc} \mathcal{O}_Y(U) & \xrightarrow{\Psi^\sharp} & \mathcal{O}_X(\Psi^{-1}U) & = & \Psi_* \mathcal{O}_X(U) \\ \Big\| & & \Big\| & & \\ \{f : U \to \mathbf{R} | f \text{ is } C^\infty\} & & \{g : \Psi^{-1}U \to \mathbf{R} | g \text{ is } C^\infty\} & & \end{array}$$

Where the map takes $f \longmapsto f \circ \Psi$.

Theorem 1.7. *Let $\Phi : A \longrightarrow B$ be a ring homomorphism. The following defines a morphism of schemes.*
$(\Psi, \Psi^\sharp) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ *where* $X =$ *Spec* (B), *and* $Y =$ *Spec* (A).

Proof. First we define $\Psi =^a \phi :$ Spec $(A) \to$ Spec (B)
Then if $[p] \in$ Spec $(B) \Rightarrow p \in B \Rightarrow \phi^{-1}p \subset A$ is a prime ideal, and by Proposition 9.7 we showed that the pullback of prime ideals is prime. That is, $A/\phi^{-1}p \overset{\phi}{\hookrightarrow} B/p$, the pullback is also a prime ideal.
Thus we make the following definition ${}^a\phi[p] := [\phi^{-1}p]$.
Now we must show that $\Psi =^a \phi$ is continuous, that is that Ψ^{-1}(closed set) is closed.
Take $J \subseteq A$ an ideal, we claim $\Psi^{-1}(V(J)) = V(\phi(J)B)$. Of course, $V(J)$ is a closed set.

$$\begin{aligned} [p] \in \Psi^{-1}(V(J)) &\Leftrightarrow \Psi([p]) \in V(J) \\ &\Leftrightarrow [\phi^{-1}p] \in V(J) \\ &\Leftrightarrow \phi^{-1}p \supset J \\ &\Leftrightarrow p \supset \phi(J) \\ &\Leftrightarrow p \supseteq \phi(J)B \end{aligned}$$

The rest of the proof will follow in lecture 29. □

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