

29. MORPHISMS II

Theorem 29.1. *Let $\varphi : A \rightarrow B$ be a ring homomorphism. This defines a morphism of schemes*

$$(\psi, \psi^\#) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y),$$

where $X = \text{Spec } B$, and $Y = \text{Spec } A$.

Proof. Define $\psi : \text{Spec } B \rightarrow \text{Spec } A$ to be the map ${}^a\varphi$ from Proposition 9.6. That is, if $[p] \in \text{Spec } B$, then $\varphi^{-1}p \subset A$ is a prime ideal and so naturally we defined

$${}^a\varphi[p] := [\varphi^{-1}p].$$

We need to see that ψ is continuous. It suffices to show that for any ideal $J \subset A$, the inverse image of $V(J)$ under ψ is closed. So consider the following:

$$\begin{aligned} [p] \in \psi^{-1}(V(J)) &\iff \psi([p]) \in V(J) \\ &\iff [\varphi^{-1}p] \in V(J) \\ &\iff \varphi^{-1}p \supset J \\ &\iff p \subset \varphi(J) \\ &\iff p \supseteq \varphi(J)B \\ &\iff p \in V(\varphi(J)B). \end{aligned}$$

Now it remains to show that $\psi^\# : \mathcal{O}_Y \rightarrow \psi_*\mathcal{O}_X$ and that the axiom holds. To do this, we need the following lemma.

Lemma 29.2. *Let $\phi : A \rightarrow B$ be a ring homomorphism and M a B -module. Let $M_{[\phi]}$ be the A -module defined by M via ϕ and $\psi = {}^a\phi$. Then as sheaves on $\text{Spec } A$ we have*

$$\psi_*(\widetilde{M}) = \widetilde{M_{[\phi]}}.$$

Proof. Let $Y = \text{Spec } A$ and $X = \text{Spec } B$. It is enough to construct isomorphisms (canonical) of sections over Y_f for each $f \in A$, $Y = \text{Spec } A$ since the Y_f cover all of Y . We need to see that $\psi^{-1}(Y_f) = X_{\phi(f)}$. To do this, recall that $Y_f = Y - V(f)$ and so

$$\psi^{-1}(Y_f) = \psi^{-1}(Y - V(f)) = \psi^{-1}(Y) - \psi^{-1}(V(f)),$$

but this is just

$$X - V(\psi^{-1}(f)) = X - V(\phi(f)) = X_{\phi(f)}.$$

Then we can calculate that

$$\psi_*(\widetilde{M})(Y_f) = \widetilde{M}(\psi^{-1}Y_f) = \widetilde{M}(X_{\phi(f)}) = M_{\phi(f)} = \left\{ \frac{m}{\phi(f)^\nu} \right\},$$

and this is the same as

$$\widetilde{M_{[\phi]}}(Y_f) = (M_{[\phi]})_f = \left\{ \frac{m}{\phi(f)^\nu} \right\}.$$

□

Now we need to define $\psi^\# : \mathcal{O}_Y \rightarrow \psi_*\mathcal{O}_X$. Again, since the Y_f cover all of Y , it is enough to define this on the Y_f . Recall that $\mathcal{O}_Y(Y_f) = A_f$. Then we see that

$$\psi_*\mathcal{O}_X = \psi_*\widetilde{B} = (\widetilde{B_{[\phi]}}),$$

where the last equality is from the lemma above. Notice also that we have

$$\psi_*\mathcal{O}_X(Y_f) = \mathcal{O}_X(\psi^{-1}Y_f) = \mathcal{O}_X(X_{\phi(f)}) = B_{\phi(f)}.$$

Then define $\psi^\# : \mathcal{O}_Y \rightarrow \psi_*\mathcal{O}_X$, by sending an element

$$\frac{a}{f^\nu} \mapsto \frac{\phi(a)}{\phi(f)^\nu}.$$

This is a homomorphism of sheaves since maximal ideals are sent to maximal ideals. \square

Scholium 29.3. A scheme is a space with a sheaf that locally looks like $(\text{Spec } A, \mathcal{O}_{\text{Spec } A})$. A morphism of schemes $f : X \rightarrow Y$ is a continuous map that is locally of the form ${}^a\phi : \text{Spec } B \rightarrow \text{Spec } A$ for a ring homomorphism $\phi : A \rightarrow B$.

Theorem 29.4. For any scheme X and any ring A , there is a one-to-one correspondence between

(1) Morphisms of schemes $X \rightarrow \text{Spec } A$

and

(2) Ring homomorphisms $A \rightarrow \mathcal{O}_X(X)$.

In other words, $\text{Hom}_{\text{sch}}(X, \text{Spec } A) \simeq \text{Hom}_{\text{rings}}(A, \mathcal{O}_X(X))$ as functors.

Proof. (1) \longrightarrow (2)

Suppose we have a morphism of schemes

$$(\psi, \psi^\#) : (X, \mathcal{O}_X) \rightarrow (\text{Spec } A, \mathcal{O}_{\text{Spec } A}),$$

where $\psi^\# : \mathcal{O}_{\text{Spec } A} \rightarrow \psi_*\mathcal{O}_X$. Taking global sections we see that we get a ring homomorphism

$$\psi^\#(\text{Spec } A) : \mathcal{O}_{\text{Spec } A}(\text{Spec } A) \rightarrow \psi_*\mathcal{O}_X(\text{Spec } A).$$

But $\mathcal{O}_{\text{Spec } A}(\text{Spec } A) = A$ and $\psi_*\mathcal{O}_X(\text{Spec } A) = \mathcal{O}(\psi^{-1}\text{Spec } A) = \mathcal{O}_X(X)$, and we get the desired ring homomorphism.

(2) \longrightarrow (1)

Suppose now that we have a ring homomorphism $\phi : A \rightarrow \mathcal{O}_X(X)$. We want to construct a morphism of schemes, and to do so we want to construct a map $\psi : X \rightarrow \text{Spec } A = Y$, and a map $\psi^\# : \mathcal{O}_Y \rightarrow \psi_*\mathcal{O}_X$. So to construct ψ , we just need to give a value in $\text{Spec } A$ for each $x \in X$. Since $\text{Spec } A$ is just the prime ideals in A , we need to find a $\mathfrak{p}_x \in A$ so that $\psi(x) = [\mathfrak{p}_x]$. Consider the map defined by the composition:

$$\phi_x : A \xrightarrow{\phi} \mathcal{O}_X(X) \xrightarrow{\lambda_x} \mathcal{O}_{X,x}.$$

Then define $[\mathfrak{p}_x] := \phi^{-1}(\mathcal{M}_{X,x})$. This is a prime ideal since it is the pullback of a maximal ideal, and then of a prime ideal. We still need to show that this is continuous and then construct the sheaf map.

For continuity, it is enough to check that $\psi|_U$ is continuous for all affine open $U \subseteq X$, since X is covered by these. So we have

$$\begin{array}{ccc} U = \text{Spec } B & \xrightarrow{i} & X \\ \psi \circ i \downarrow & \swarrow \psi & \\ V(J) \subset \text{Spec } A & & \end{array}$$

where i is the inclusion map and $V(J)$ is a closed subset of $\text{Spec } A$. So we see that $U \cap \psi^{-1}(V(J)) \subset U$ is closed. Consider the map

$$\phi_0 : A \xrightarrow{\phi} \mathcal{O}_X(X) \xrightarrow{\text{restriction}} \mathcal{O}_X(U) = B.$$

We need to check that $\psi \circ i = {}^a\phi_0$. To do this, take $x \in U$. Then $\mathcal{O}_{X,x} = \mathcal{O}_{U,x} = B_{\mathfrak{q}}$ if $x = [\mathfrak{q}]$, where $\mathfrak{q} \subset B$. Then $\mathcal{M}_{X,x} = \mathfrak{q} \cdot B_{\mathfrak{q}}$. Recall that $\mathfrak{p}_x = \phi_x^{-1}(\mathcal{M}_{X,x})$. We claim that $[\mathfrak{p}_x] = [\phi_0^{-1}\mathfrak{q}] = {}^a\phi_0[\mathfrak{q}]$. We have the following diagram:

$$\begin{array}{ccc} A & \xrightarrow{\phi} & \mathcal{O}_X(X) \\ & \searrow \phi_0 & \downarrow \text{restr.} \\ & & \mathcal{O}_X(U) = B \\ & \searrow \phi_x & \downarrow \text{loc.} \\ & & \mathcal{O}_{X,x} = B_{\mathfrak{q}} \end{array}$$

Recall this fact about localizations: if $B \xrightarrow{\text{loc.}} B_{\mathfrak{q}} \supset \mathfrak{q} \cdot B_{\mathfrak{q}}$, then $\mathfrak{q} = \text{loc}^{-1}\mathfrak{q}B$. Now we have

$$(\psi \circ i)[\mathfrak{q}] = [(\text{loc} \circ \phi_0)^{-1}(\mathfrak{q}B_{\mathfrak{q}})] = [\phi_0^{-1}(\text{loc}^{-1})(\mathfrak{q}B_{\mathfrak{q}})] = [\phi_0^{-1}\mathfrak{q}] = {}^a\phi_0[\mathfrak{q}].$$

We still need to construct $\psi^{\#}$, and this will be continued next class. \square