

## LECTURE 3

MATTHEW BENNETT

### 1. V-I CORRESPONDENCE

Recall that the following correspondence exists.

$$\left\{ \begin{array}{l} \text{ideals} \\ J \subseteq k[x_1, \dots, x_m] \end{array} \right\} \begin{array}{c} \xrightarrow{V} \\ \xleftarrow{I} \end{array} \left\{ \begin{array}{l} \text{subsets} \\ X \subseteq \mathbb{A}(k) \end{array} \right\}$$

**Proposition 1.1.** *Fundamental Properties of the V-Correspondence*

- (i)  $V(0) = \mathbb{A}^m$  and  $V(A) = \emptyset$  where  $A = k[x_1, \dots, x_n]$ ,  $k$  a field.
- (ii)  $J_1 \subseteq J_2 \Rightarrow V(J_1) \supseteq V(J_2)$
- (iib)  $\sqrt{J_1} \subseteq \sqrt{J_2} \Rightarrow V(J_1) \supseteq V(J_2)$
- (iii)  $\cup_{j=1}^r V(J_j) = V(\cap_{i=1}^r J_i) = V(J_1 J_2 \dots J_r)$
- (iv)  $\cap_{\lambda \in \Lambda} V(J_\lambda) = V(\sum_{\lambda \in \Lambda} J_\lambda)$

Statements (i),(ii), and (iv) are trivial matters, but we will prove statement (iii).

*Proof of 1.1 (iii).* We prove (iii) by using induction on  $r$ . If  $r = 1$  then the statement is trivial. It is enough to show the case for when  $r = 2$ , that is, show  $V(J_1) \cup V(J_2) = V(J_1 \cap J_2) = V(J_1 J_2)$ . It is necessary and sufficient to prove the following statement

$$V(J_1) \cup V(J_2) \subseteq^* V(J_1 \cap J_2) \subseteq^{**} V(J_1 J_2) \subseteq^{***} V(J_1) \cup V(J_2).$$

*Proof of \*\*.* Note that  $J_1 J_2 \subseteq J_1 \cap J_2$ , which implies by 1.1 (iia) that

$$V(J_1 \cap J_2) \subseteq V(J_1 J_2).$$

□

*Proof of \*.* Let  $x \in V(J_1) \cup V(J_2)$ . Then either  $g(x) = 0$  for all  $g$  in  $J_1$ , or  $g(x) = 0$  for all  $g$  in  $J_2$ . Now let  $f \in J_1 \cap J_2$ . Then  $f \in J_1$  and  $f \in J_2 \Rightarrow f(x) = 0 \Rightarrow x \in V(J_1 \cup J_2)$  since the statement is true for any  $f$ . □

*Proof of \*\*\*.* Here we prove the contrapositive. We will show if  $x \notin V(J_1) \cup V(J_2) \Rightarrow x \notin V(J_1 J_2)$ . Since  $x \notin V(J_1) \cup V(J_2)$ , then  $x \notin V(J_1)$  and  $x \notin V(J_2)$ . Thus there exists an  $f_1 \in J_1$  with  $f_1(x) \neq 0$  and there exists an  $f_2 \in J_2$  with  $f_2(x) \neq 0$ . Then since  $f_1 f_2 \in J_1 J_2$  and  $(f_1 f_2)(x) = f_1(x) f_2(x) \neq 0$ , which is true since we are in a field. Therefore,  $x \notin V(J_1 J_2)$ . □

---

*Date:* August 26, 2005.

□

**Definition 1.2.** The Zariski topology on  $\mathbb{A}^n(k)$  is the topology whose closed sets are  $V(J)$ , for ideals  $J \subseteq k[x_1, \dots, x_n]$ .

By 1.1, the Zariski topology is a topology. In fact, it is a very coarse topology.

**Example 1.3.** Take  $\mathbb{A}^1(\mathbb{C})$ . Then the ideals,  $J$  are principal, that is,  $J = \langle f \rangle \subset k[x]$ . Implying

$$V(J) = \{a \in \mathbb{A}^1(k) : f(a) = 0\}.$$

Thus  $f(x) = c(x - a_1)(x - a_2) \dots (x - a_m)$ . Therefore  $V(f) = \{a_1, \dots, a_m\}$ . Notice that the closed sets are the empty set, the whole space, and the finite sets. Also the open sets are the empty set, the whole space, and the complements of the finite sets.

**Definition 1.4.** If  $X \subseteq \mathbb{A}^n(k)$ , then  $I(X) = \{f \in k[x_1, \dots, x_m] : f|_X \equiv 0\}$ , which is an ideal.

**Proposition 1.5.** *Properties of the I-Correspondence*

1.  $I(\emptyset) = k[x_1, \dots, x_m]$  and  $I(\mathbb{A}^n) = (0)$ .
2.  $X_1 \subseteq X_2 \Rightarrow I(X_1) \supseteq I(X_2)$ .
3.  $X \subseteq V(I(X))$  with equality if and only if  $X$  is an algebraic set. In general,  $V(I(X)) = \bar{X}$  where  $\bar{X}$  is the closure of  $X$  in the Zariski topology.
4.  $J \subseteq I(V(J))$ . The stronger statement  $\sqrt{J} \subseteq I(V(J))$  is also true.
5.  $NSZ : I(V(J)) = \sqrt{J}$ .

*Proof of part (3) of 1.5.* Let  $W \supset X$  be a closed set. Then  $W = V(J)$  by definition. Then  $V(J) \supset X \Rightarrow I(V(J)) \subseteq I(X)$ . Then by part 4 of the proposition,  $J \subseteq I(V(J)) \subset I(X)$ . Thus  $V(J) = W \supseteq V(I(X))$ . Therefore, for all closed sets  $W \supset X$ , we have that  $W \supset V(I(X))$ , and it follows that since  $X \subseteq \bar{X} \Rightarrow \bar{X} \supseteq V(I(X))$ . Now we prove the other containment. By definition  $V(I(X))$  is closed, thus if  $X \subseteq V(I(X))$ , then  $\bar{X} \subseteq V(I(X))$ . Combining both containments implies  $V(I(X)) = \bar{X}$ . □

**Definition 1.6.** A nonempty subset  $Y \subseteq X$  in a topological space is called irreducible if whenever  $Y = Y_1 \cup Y_2$  with  $Y_1, Y_2 \subseteq Y$  both closed, implies  $Y = Y_1$  or  $Y = Y_2$ .

**Lemma 1.7.** *If  $X \subseteq \mathbb{A}^n(k)$  is an algebraic set, the  $X$  is irreducible if and only if  $I(X)$  is prime.*

*Proof.* We will prove the contrapositive. If  $X$  is not irreducible, then  $I(X)$  is not prime.

( $\Rightarrow$ ) Suppose  $X$  is reducible, that is,  $X = X_1 \cup X_2$  where  $X_1$  and  $X_2$  are nonempty. Then  $X_1 \subsetneq X \Rightarrow I(X_1) \supsetneq I(X)$ . We show that  $I(X_1) \neq I(X)$  by supposing that if  $I(X_1) = I(X) \Rightarrow V(I(X_1)) = V(I(X))$ , but  $V(I(X_1)) =$

$\bar{X}_1 = V(I(X)) = X$ , which is a contradiction. Thus  $I(X_1) \supsetneq I(X)$ . Similarly,  $I(X_2) \subsetneq I(X)$  implying that there exists an  $f_1 \in I(X_1)/I(X)$  where  $f_1|_{X_1} \equiv 0$  and there exists an  $f_2 \in I(X_2)/I(X)$  with  $f_2|_{X_2} \equiv 0$ . Then

$$f_1 f_2|_{X_1 \cup X_2} \equiv 0 \Rightarrow f_1 f_2 \in I(X)$$

Therefore  $I(X)$  is not prime.

( $\Leftarrow$ ) Now suppose  $I = I(X)$  is not prime. Then there exists  $f_1$  and  $f_2 \notin I(X)$  such that  $f_1 f_2 \in I(X)$ . Let  $I_1 = (I, f_1)$ , the ideal generated by  $I$  and  $f_1$ , and let  $I_2 = (I, f_2)$ . Set  $V(I_1) = X_1$  and  $V(I_2) = X_2$ . Here we note that  $I \subsetneq I_1$  and  $I \subsetneq I_2$ . Thus we get the following,  $X_1 = V(I_1) \subsetneq V(I) = V(I(X)) = X$ . Similarly  $X_2 = V(I_2) \subsetneq V(I) = V(I(X)) = X$ . Thus  $X_1, X_2 \subsetneq X$ . Now we need to show  $X_1 \cup X_2 = X$ . Clearly  $X_1 \cup X_2 \subseteq X$ . Need to show  $X_1 \cup X_2 \supseteq X$ . We have  $X = V(I(X)) = V(I)$  by part 3 of proposition 1.5 above. Thus  $x \in X$  if and only if  $f(x) = 0$  for all  $f \in I$ . Because  $f_1 f_2 \in I(X)$  we have  $f_1(x) f_2(x) = 0$ , and so either  $f_1(x) = 0$  or  $f_2(x) = 0$ , and hence either  $x \in V(I_1) = X_1$  or  $x \in V(I_2) = X_2$ . We have written  $X$  as a union of closed subset in a nontrivial way so  $X$  is reducible.  $\square$

LOUISIANA STATE UNIVERSITY, DEPARTMENT OF MATHEMATICS, BATON ROUGE, LA 70803, USA

*E-mail address:* mtbnit@math.lsu.edu