32. Categories and Functors

Definition 32.1. A *category* consists of the following data:

(1) A class of sets A, B, C, ... *etc.*
(2) A class of *morphisms f, g, h, ... etc.* Each morphism has a unique *domain* and *codomain*, which are objects of the category. We write $f : A \to B$ to mean that the domain of f is A and the codomain is B.
(3) A composition rule $(f, g) \to fg$ that assigns to any pair $g : A \to B$ and $f : B \to C$ an element $fg : A \to C$. This rule must be associative, i.e. $(fg)h = f(gh)$ whenever the compositions are defined; also, for each object A there must be an identity $id_A : A \to A$ which satisfies $fid_A = f$ and $id_A g = g$ whenever the compositions are defined.

We typically use boldface letters **C, D, ...** etc. to stand for categories. **C** is said to be *locally small* if for each pair of objects A, B, the class of morphisms $f : A \to B$ is a set (rather than a proper class). Most categories normally encountered are locally small. Some people take this to be part of the definition of a category – and we, too, will follow this convention.

Definition 32.2. A *functor* F from category **C** to category **D** consists of a map $A \to F(A)$ from the objects of **C** to the objects of **D** and a map $f \to F(f)$ from the morphisms of **C** to the morphisms of **D** such that:

- if $f : A \to B$, then $F(f) : F(A) \to F(B)$;
- $F(id_A) = id_{F(A)}$;
- $F(fg) = F(f)F(g)$.

Example 32.3.

a: *The category* ***Set*** *has as objects all sets. If* A *and* B *are sets,* $hom_{\textbf{Set}}(A, B)$ *is the set of all functions from* A *to* B.
b: *The category* ***Grp*** *has as objects all groups. If* A *and* B *are groups,* $hom_{\textbf{Grp}}(A, B)$ *is the set of all homomorphisms from* A *to* B.
c: *The so-called forgetful functor* $F : \textbf{Grp} \to \textbf{Set}$ *is defined by letting* $F(A)$ *be the underlying set of* A *and letting* $F(f)$ *be the function of sets underlying the homomorphism* f.
d: *The free group functor* $G : \textbf{Set} \to \textbf{Grp}$ *is defined by letting* $G(X)$ *be the free group on the set* X. $G(f) : G(X) \to G(Y)$ *is defined to be the homomorphism induced by the set function* $f : X \to Y$. *(Note that we may regard* X *(respectively* Y*) as a subset of (the underlying set of)* $G(X)$ *(respectively* $G(Y)$*). So* $f : X \to Y$ *provides a (set) function from the free generators of* $G(X)$ *to* $G(Y)$*, and by the definition of free group, this determines a unique morphism from* $G(X)$ *to* $G(Y)$.

Definition 32.4. In a category **C**, a morphism $f : A \to B$ is said to be an *isomorphism* if there is a morphism $g : B \to A$, called the inverse of f, such that $fg = id_B$ and $gf = id_A$.

Definition 32.5. Let **C** be a category, I a set, and $\{A_i | i \in I\}$ an indexed family of objects in **C**. An object P together with a family of morphisms $p_i : P \to A_i$ $i \in I$ in **C** is called a (*categorical*) *product* (*of the* A_i*s*) if the following condition holds: For any family $f_i : T \to A_i$ $(i \in I)$ of morphisms, there is a unique morphism $f : T \to P$ such that for all $i \in I$, $f_i = p_i f$.

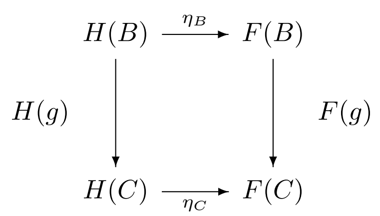
Given a morphism $f : A \to B$ of $\mathbf{C}$, then the rule $\gamma \to \gamma f$ induces a function from $hom_{\mathbf{C}}(B, T)$ to $hom_{\mathbf{C}}(A, T)$. Note that the induced function goes in the "wrong"' direction. There are two ways to deal with it.
1) We can make from $\mathbf{C}$ a new category $\mathbf{C}^{op}$, which has the same objects as $\mathbf{C}$ but whose morphisms are defined by $hom_{\mathbf{C}^{op}}(A, B) := hom_{\mathbf{C}}(B, A)$. The composition rule $\circ^{op}$ of $\mathbf{C}^{op}$ is defined by $f \circ^{op} g := g \circ f$, where $\circ$ is the composition rule of $\mathbf{C}$.
2) We can define the concept of a contravariant functor in analogy with the functor above.

Natural Transformations

Suppose U and V are objects of $\mathbf{C}$ and $t : U \to V$ is a morphism. We have two fuctors $hom_{\mathbf{C}}(U,{}_{)}$ and $hom_{\mathbf{C}}(V, *)$. To save space, we denote these F and H, respectively. What kind of relationship does the morphism t induce between these functors? First, note that for any $\mathbf{C}$-object A, we get a set fuction $\tau_A : H(A) \to F(A)$ by the rule $\tau_A(a) := at$. Second, note that if $f : A \to B$ is a morphism of $\mathbf{C}$, then $F(f)\tau_A = \tau_B H(f)$, since if $a \in H(A)$,

$$F(f)(\tau_A(a)) = f(at) = (fa)t = \tau_B(H(f))(a).$$

Definition 32.6. Suppose that H and F are functors from $\mathbf{C}$ to $\mathbf{D}$. A *natural transformation* η from H to F is a collection of $\mathbf{D}$-morphisms $\eta_B : H(B) \to F(B)$, one for each object B of $\mathbf{C}$, such that for any $g : B \to C$ in $\mathbf{C}$, $F(g)\eta_B = \eta_C H(G)$. In diagrams:



If every η_B is an isomorphism, we say that η is a *natural isomorphism.*

Lemma 32.7. *(Yoneda's Lemma) Let F be a functor from $\mathbf{C}$ to **Set** and let A be an object of $\mathbf{C}$.*

(1) *For each $a \in F(A)$ there is a natural transformation $\eta(a)$ from $hom_{\mathbf{C}}(A, \_\_)$ to F whose component at B, $a_B : hom_{\mathbf{C}}(A, B) \to F(B)$, is defined by $a_B(k) = F(k)(a)$ $(k \in hom_{\mathbf{C}}(A, B))$.*
(2) *$a \to \eta(a)$ is a bijection of $F(A)$ with the class of natural transformations from $hom_{\mathbf{C}}(A, \_\_)$ to F. The inverse is $\eta \to \eta_A(id_A) \in F(A)$.*

Proof. Let $g : B \to C$ be a morphism of $\mathbf{C}$. For the sake of notation, let H_A denote the functor $hom_{\mathbf{C}}(A, \_\_)$. Then for any $k : A \to B$,

$$F(g)a_B(k) = F(g)F(k)(a) = F(gk)(a)$$

$$a_C H_A(g)(k) = a_C(gk) = F(gk)(a).$$

Thus, (1) is proved. To prove (2), it suffices to show that the maps mentioned are inverses of one another. First, pick any $a \in F(A)$. Then

$$\eta(a)_A(id_A) = a_A(id_A) = F(id_A)(a) = a.$$

Next, let η be any natural transformation from H_A to F. Let $a := \eta_A(id_A) \in F(A)$. We must show that $\eta(a) = \eta$. That is, we must show that $\eta(a)_B = \eta_B$ for all objects B, i.e., for any B and any $f \in \hom_{\mathbf{C}}$,

$$\eta(a)_B(f) = \eta_B(f).$$

Now, we get

$\eta_B(f) = \eta_B(f\ id_A) = \eta_B(H_A(f)(id_A)) = F(f)\eta_A(id_A) = F(f)a = a_B(f) = \eta(a)_B(f)$.

□

Definition 32.8. Let F :

Corollary 32.9. *If F is representable, the object representing F is unique up to isomorphism.*

Example of a representable functor:

Example 32.10. *Let $\boldsymbol{C}$ be the category of groups and $f : G \to H$ a homomorphism. Define a functor $F^f : \boldsymbol{C} \to \boldsymbol{Sets}$ where*
$F^f(L) = \{\alpha : F \to G | f \circ \alpha = \boldsymbol{C}_H\}$.
There exists a pair (K, a_K), K a group and
$a_k \in F^f(K) = \{a_k : K \to G | f \circ a_K = \boldsymbol{C}_H\}$. This is the kernel of f;
$Ker(f) = \{x \in G | f(x) = \boldsymbol{C}_H\}$. $K = Ker(f)$, $a_k : Ker(f) \hookrightarrow G$.