

4. NULLSTELLENSATZ

These are notes from September 9, 2005, transcribed by Charles Egedy. The literal translation of *Nullstellensatz*, abbreviated NSZ, is “theorem about location of zeroes”

Theorem 4.1. *Let k be an algebraically closed field. The following are equivalent and true:*

- (a) *Every maximal ideal of $A = k[x_1, \dots, x_n]$ is of the form $M_a = \langle x_1 - a_1, \dots, x_n - a_n \rangle$, where $a \in k^n$.*
- (b) *For any ideal $J \subsetneq A$, $V(J) \neq \emptyset$.*
- (c) *For any ideal $J \subsetneq A$, $I(V(J)) = \sqrt{J}$.*

Proof. Of Equivalence:

(a \Rightarrow b) Let J be a proper ideal in A . Then J is contained in a maximal ideal M . Let $J \subseteq M \subset A$. By (a), $M = M_a$ for some $a \in k^n$, so $J \subseteq M_a$. Observe that $a \in V(J)$, since $f \in J$ can be written in the form $\sum_{i=1}^n q_i(x)(x_i - a_i)$, and so $f(a) = 0$.

(b \Rightarrow a) Let $M \subsetneq A$ be a maximal ideal, and note that $V(M) \neq \emptyset$ by assumption. Taking $a \in V(M)$, we show that $M = M_a$. Define a map $\phi_a : A \rightarrow k$ by $\phi_a(f) = f(a)$. By its definition, $M_a \subseteq \ker(\phi_a)$. Now for any $f(x) \in A$ we can express $f(x)$ using Taylor’s formula. Thus $f(x) = f(a) + \sum q_i(x_i - a_i) +$ “higher order terms”. But if $f(x) \in \ker(\phi_a)$, we have that $f(a) = 0$ and so $f(x) \in M_a$. Therefore $M_a \subseteq \ker(\phi_a)$, and so $M_a = \ker(\phi_a)$. Thus we have shown that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\phi_a} & k \\ \text{proj} \downarrow & & \parallel \\ A/M_a & \xrightarrow{\bar{\phi}_a} & k \end{array}$$

In particular ϕ_a is surjective, and so A/M_a is a field (isomorphic to k) and so M_a is maximal. Hence every $f \in M$ is also in M_a and so $M \subseteq M_a$. However, M is maximal and so $M = M_a$.

(b \Rightarrow c) We have already shown that $\sqrt{J} \subseteq I(V(J))$, and so we need just the other inclusion. Take $f \in I(V(J))$, so $f|_{V(J)} = 0$. We need to show that $f^N \in J$ for some $N \geq 1$. (The following tool is due to Rabinowitch.)

Consider a new ring $B = A[T] = k[x_1, \dots, x_n, T]$, and an ideal $J_1 \subseteq B$ generated by J and by $Tf - 1$, (so $J_1 = \langle J, Tf - 1 \rangle$). We claim that $V(J)$ is empty in $k^n \times k$, for if $g(a, t) = 0 \forall g \in J_1$, then $g(a) = 0 \forall g \in J$, and $tf(a) - 1 = 0$; however this last equation cannot hold as $f(a) = 0$. So J_1 must be all of B by (b). Then $1 \in J_1$. By the Hilbert Basis Theorem, $J = \langle g_1, \dots, g_s \rangle$. Thus $1 = \sum_{i=1}^s h_i(x, T)g_i(x) + h_0(x, T)(Tf(x) - 1)$. Now substitute $T = 1/f(x)$, so the last term is identically zero. The expression obtained is not a polynomial expression; however, we can clear all rational terms by multiplying by f^N for N sufficiently large. Thus we have $f^N = \sum [\text{polynomial}]_i g_i(x)$, so $f^N \in J$, meaning that $f \in \sqrt{J}$.

(c \Rightarrow b) Let $J \subsetneq A$ and suppose that $V(J) = \emptyset$. Applying (c) we find that $\sqrt{J} = A$. This means that every polynomial in A when raised to a sufficiently high integral power is in J . Therefore, $1 \in J$ and so $J = A$, contradicting the assumption that $J \subsetneq A$, and so $V(J)$ cannot be empty. \square

We need for the field to be algebraically closed in the above proof. Consider for example $f \in \mathbb{R}[x, y]$, where $f = x^2 + y^2 + 1$. Note that $V(f) \subseteq \mathbb{R}$ is empty, so if the theorem applied, the ideal generated by f should be the entire ideal, a false conclusion.

The maps V and I gave us a correspondence between ideals of J in A and algebraic subsets of $\mathbb{A}^n(k)$. We can write the following corollary to the NSZ:

Corollary 4.2. *The NSZ gives the following correspondences, noting containment as one moves down the list of sets to the left of the arrows and to the right of the arrows:*

$$\begin{aligned} \{\text{radical ideals } J = \sqrt{J}\} & \xleftrightarrow{\text{one-to-one}} \{\text{algebraic subsets}\} \\ \{\text{prime ideals } P \subset A\} & \xleftrightarrow{\text{one-to-one}} \{\text{irreducible algebraic sets}\} \\ \{\text{maximal ideals } M \subset A\} & \xleftrightarrow{\text{one-to-one}} \{\text{points in } \mathbb{A}^n(k)\} \end{aligned}$$

Proof. \square

Thus the point (a, b) in affine space corresponds to the ideal $\langle x - a, y - b \rangle$. We write that $(a, b) \in V(f) \Leftrightarrow \langle f \rangle \subseteq \langle x - a, y - b \rangle$.