LECTURE 7

PIOTR MACIAK

ABSTRACT. This lecture provides some examples that ilustrate the theory we have developed so far.

1. POINTS IN $\mathbb{A}^n$

Proposition 1.1. *Any finite set of points in $\mathbb{A}^n$ is an affine algebraic set.*

Proof. Let $a = (a_1, \ldots, a_n) \in \mathbb{A}^n$ and $M_a = \langle x_1 - a_1, \ldots, x_n - a_n \rangle$. Clearly, $V(M_a) = \{a\}$. Thus, if $A = \{a^1, a^2, \ldots, a^s\}$ is a finite set of points in $\mathbb{A}^n$, then

$$A = \bigcup_{i=1}^{s} \{a^i\} = \bigcup_{i=1}^{s} V(M_{a_i}) = V(\bigcap_{i=1}^{s} M_{a^i}).$$

□

Corollary 1.2. $I(\{a^1, a^2, \ldots, a^s\}) = \bigcap_{i=1}^{s} M_{a^i}$.

Proof. It follows immediately from Nullstelensatz that $I(\{a^1, a^2, \ldots, a^s\})$ equals the radical of $\bigcap_{i=1}^{s} M_{a^i}$. We have seen in Lecture 4 that for every $a \in \mathbb{A}^n$, the ideal M_a is maximal and hence prime. In order to complete the proof, it is enough to solve the following exercise. □

Exercise 1.3. *Any finite intersection of prime ideals is a radical ideal.*

Proof. Let $\mathfrak{p}_1$, $\mathfrak{p}_2, \ldots, \mathfrak{p}_s$ be prime ideals. It is clear that $\bigcap_{i=1}^{s} \mathfrak{p}_i$ is an ideal. We shall show that it is radical. Let $x^m \in \bigcap_{i=1}^{s} \mathfrak{p}_i$. Then for each i, $x^m \in \mathfrak{p}_i$. Since $\mathfrak{p}_i$ is prime, it follows that $x \in \mathfrak{p}_i$. Thus, $x \in \bigcap_{i=1}^{s} \mathfrak{p}_i$. □

Remark 1.4. In a Noetherian ring, the converse is true.

2. PLANE CURVES $C \subset \mathbb{A}^2$

Definition 2.1. We define the *dimension* of an algebraic set X to be Krull dimension of its affine ring.

Definition 2.2. An *affine curve* is an affine variety of dimension 1.

Remark 2.3. Since we have not developed the dimension theory properly and in this section we are interested mainly in plane curves $C \subset \mathbb{A}^2$, we will work with the following simpler definition.

Definition 2.4. Let $f(x, y) \in k[x, y]$ be an irreducible polynomial. Since $k[x, y]$ is a UFD, it follows that $\langle f \rangle$ is a prime ideal. Consequently, $V(f)$ is an affine variety. We call it the *affine curve* defined by f.

Example 2.5. Let $f(x, y) = y^2 - x$. First, we show that $f(x, y)$ is irreducible. Suppose that $f(x, y) = g(x, y) \cdot h(x, y)$. Since f, seen as an element of $k[x][y]$, is monic and of degree 2, we may assume that $g(x, y) = y - a(x)$ and $h(x, y) = y - b(x)$ for some polynomials $a(x)$, $b(x) \in k[x]$. But then

$$a(x)^2 - x = f(x, a(x)) = g(x, a(x)) \cdot h(x, a(x)) = 0,$$

which yields $a(x)^2 = x$. This is a contradiction for $\deg a(x)^2$ is even, while $\deg x = 1$. Thus, $f(x, y)$ must be irreducible. Consequently, $\langle f \rangle$ is prime and $C = V(f)$ is an affine curve. Since every prime ideal is radical, it follows from Nullstelensatz that $I(C) = I(V(f)) = \langle f \rangle$. Now, let $\psi : k[x, y] \to k[x]$ be given by $\psi(g(x, y)) = g(x, x^2)$. Clearly, ψ is an epimorphism of k-algebras. The kernel of ψ equals $I(C)$. Indeed,

$$\begin{aligned} \ker \psi &= \{\, g(x, y) \in k[x, y] \mid g(x, x^2) = 0 \,\} \\ &= \{\, g(x, y) \in k[x, y] \mid g_{|C} = 0 \,\} \\ &= I(C). \end{aligned}$$

Thus,

$$k[C] = k[x, y]/I(C) \cong k[y].$$

We have seen in Lecture 5 that affine algebraic are isomorphic if and only if their coordinate rings are isomorphic as k-algebras. Since $k[y]$ is the coordinate ring of $\mathbb{A}^1$, it follows that the curve C is isomprphic to the affine line $\mathbb{A}^1$.

Example 2.6. Now, we shall give an example of a bijective morphism which is not an isomorphism. Let $f(x, y) = y^2 - x^3$. First, we show that $f(x, y)$ is irreducible. Repeating the argument we used in Example 2.5, we see that reducibility of f implies that for some $a(x) \in k[x]$, we have $a(x)^2 = x^3$. Since $\deg a(x)^2$ is even and $\deg x^3 = 3$, it gives a contradiction and we conclude that f is irreducible. Consequently, $C(f)$ is an affine curve. The curve C is called *cubic cusp.* Since every prime ideal is radical, it follows from Nullstelensatz that $I(C) = I(V(f)) = \langle f \rangle$.

Now, consider the morphism $\varphi : \mathbb{A}^1 \to \mathbb{A}^2$ defined by $\varphi(t) = (t^2, t^3)$. We claim that φ induces a bijective morphism ϕ between $\mathbb{A}^1$ and C

which is not an isomorphism. Clearly, $\varphi(\mathbb{A}^1) \subset C$. On the other hand, choose $(x, y) \in C$, that is $(x, y) \in k^2$ with $y^2 = x^3$. Since k is algebraically closed, there is $a \in k$ such that $a^2 = (-a)^2 = x$. Then

$$(a^3)^2 = (a^2)^3 = x^3 = y^2,$$

which follows that either $a^3 = y$ or $(-a)^3 = -a^3 = y$. Consequently, either $\varphi(a) = (x, y)$ or $\varphi(-a) = (x, y)$, which means that φ maps $\mathbb{A}^1$ onto C. Now, we shall show that φ is 1-1. Suppose that φ is not 1-1. Then $(t^2, t^3) = (s^2, s^3)$ for some $t \neq s$. Since $t^2 = s^2$, it follows that $t = -s$. But then $s^3 = t^3 = (-s)^3 = -s^3$, which follows that $t = s = 0$. A contradiction. Hence if $\phi : \mathbb{A}^1 \to C$ is given by $\phi(t) = \varphi(t) = (t^2, t^3)$, then ϕ is a bijective morphism between $\mathbb{A}^1$ and C. Now, let $\varphi^* : k[x, y] \to k[t]$, and $\phi^* : k[C] \to k[t]$ be homomorphisms of k-algebras that correspond [1] to morphisms $\varphi : \mathbb{A}^1 \to \mathbb{A}^2$, $\phi : \mathbb{A}^1 \to C$. Then $\varphi^*(x) = t^2$, $\varphi^*(y) = t^3$ and $\varphi^* = \phi^* \circ \pi$, where $\pi : k[x, y] \to k[C]$ is the canonical projection. In order to prove that ϕ is not an isomorphism, it is enough to show that ϕ^* is not surjective. If $g(x, y) = \sum a_{ij} x^i y^j$, then $\phi^*(g + \langle f \rangle) = \sum a_{ij} t^{2i} t^{3j}$. Thus, if $h(t) \in \mathrm{Im}(\phi^*)$, then either h is constant or $\deg h(t) \leq 2$. In particular, t is not a value of ϕ^* and hence ϕ is not an isomorphism.

Example 2.7. According to Definition 2.2, an affine curve does not need to be a subset of $\mathbb{A}^2$. Indeed, if $C = \{ (t, t^2, t^3) \mid t \in k \}$, then one can show that C is an affine curve. The curve C is called *twisted cubic.*

References

[1] Hartshorne, R., *Algebraic Geometry* , Springer-Verlag., New York, 1977.

Mathematics Department, Louisiana State University, Baton Rouge, Louisiana
E-mail address: pmaciak@math.lsu.edu

[1]We have seen in Lecture 5 that if X, Y are affine algebraic sets, then there is a canonical bijection between the set of morphisms $\mathrm{Mor}(X, Y)$ and the set of homomorphisms of k-algebras $\hom_k(k[Y], k[X])$.