Cohomology of the Siegel modular group of degree two and level four

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Abstract

We compute the cohomology of the subgroup of the integral symplectic group of degree 2 consisting of matrices $\gamma \equiv \mathbf{1} \bmod 4$. This is done by computing the cohomology of the moduli space of principally polarized abelian surfaces with a level 4 structure. This space has a projective smooth compactification whose topology was analyzed by Lee and Weintraub in a recent work.

1 Introduction

The integral symplectic group

$$\mathbf{Sp}_{2d}(\mathbf{Z}) \;=\; \{\gamma \in \mathbf{M}_{2d}(\mathbf{Z}) : {}^t\gamma J \gamma = J\} \;\; \text{where } J = \begin{pmatrix} 0 & 1_d \\ -1_d & 0 \end{pmatrix}$$

operates on Siegel's upper half space of degree d

$$\mathfrak{S}_d \;=\; \{\tau \in \mathbf{M}_d(\mathbf{C}) : {}^t\tau = \tau,\ \mathrm{Im}(\tau) > 0\}$$

via

$$\gamma.\tau \;=\; (A\tau + B)(C\tau + D)^{-1}, \;\; \gamma \;=\; \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

For subgroups of finite index $\Gamma \subset \mathbf{Sp}_{2d}(\mathbf{Z})$ one is interested in the quotient space

$$X_\Gamma \;=\; \Gamma \backslash \mathfrak{S}_d$$

Recall that

1. These spaces are quasi - projective algebraic varieties of (complex) dimension $d(d+1)/2$, and they are smooth if Γ is torsion - free, which is the case for all sufficiently small Γ. We will refer to these varieties as Siegel modular varieties. These admit compactifications to projective varieties, in a number of ways, using the mechanism of toroidal embeddings. Denote these compactifications by X_Γ^*. These are smooth if Γ has no torsion and the complement $X_\Gamma^* - X_\Gamma$ is a divisor with normal crossings. For $d = 2$ there is a canonical one, first constructed by Igusa

2. The X_Γ are moduli spaces of principally polarized abelian varieties with a certain type of a rigidification called a level structure.

3. There is a canonical isomorphism

$$\mathrm{H}^*(X_\Gamma,\ \mathbf{Q}) \ = \ \mathrm{H}^*(\Gamma,\ \mathbf{Q})$$

   which holds even with $\mathbf{Z}$ - coefficients if Γ is torsion - free.

The case $d = 1$ are the modular curves, whose topology was worked out in the nineteenth century, and whose arithmetic properties are a topic of current research. One would like to compute these groups for $d \geq 2$. Every such Γ contains a congruence subgroup

$$\Gamma_d(n) \ = \ \{\gamma \in \mathbf{Sp}_{2d}(\mathbf{Z}) \ : \ \gamma \equiv \ \mathbf{1} \bmod n\}$$

of some level n (these are torsion - free for $n \geq 3$). Sometimes we omit the suffix d if it is clear. We set

$$X_{\Gamma(n)} \ = \ \mathcal{A}_d(n) \text{ and } X^*_{\Gamma(n)} \ = \ \mathcal{A}_d(n)^*$$

to emphasize the relation with abelian varieties. Notice that the finite group

$$\mathbf{Sp}_{2d}(\mathbf{Z})/\Gamma(n) \ = \ \mathbf{Sp}_{2d}(\mathbf{Z}/n)$$

operates on $\mathcal{A}_d(n)$ and on $\mathcal{A}_d(n)^*$, so that if say $\Gamma_d(n) \subset \Gamma$ then

$$\mathrm{H}^*(\Gamma,\ \mathbf{Q}) \ = \ \mathrm{H}^*(\Gamma_d(n),\ \mathbf{Q})^{\Gamma/\Gamma_d(n)}$$

Therefore it is sufficient to understand the groups $\mathrm{H}^*(\Gamma_d(n),\ \mathbf{Q})$ as modules over $\mathbf{Sp}_{2d}(\mathbf{Z}/n)$. Very few of these have been computed. In fact, apart from the case $d = 1$, complete results exist ($\mathbf{Q}$ - coefficients) only for $d = 2$,

$n = 1, 2$ by Lee and Weintraub [17], [18], and recently $d = 2$, $n = 3$ by Hoffman and Weintraub [11]. This paper treats the case $d = 2$, $n = 4$. From now on, $d = 2$ unless explicitly mentioned otherwise. Actually, the cohomology of the compactified space $\mathcal{A}_2(4)^*$ was determined by Lee and Weintraub, so the problem is to use this information to get the cohomology of the open subset $\mathcal{A}_2(4)$ This is done by studying the Leray spectral sequence for the inclusion

$$j : \mathcal{A}_2(4) \hookrightarrow \mathcal{A}_2(4)^*$$

As Deligne observed, the filtration defined by this spectral sequence is the weight filtration for the mixed Hodge structure on $\mathrm{H}^*(\mathcal{A}_2(4), \mathbf{Q})$. Let us recall the main theorem of [18]:

Theorem 1.1. *The Hodge numbers* $h^{p,q} = \dim_{\mathbf{C}} H^{p,q}(\mathcal{A}_2(4)^*)$ *are*

1. $\mathrm{h}^{0,0} = \mathrm{h}^{3,3} = 1$
2. $\mathrm{h}^{0,1} = \mathrm{h}^{1,0} = \mathrm{h}^{3,2} = \mathrm{h}^{2,3} = 0$
3. $\mathrm{h}^{0,3} = \mathrm{h}^{3,0} = 15$
4. $\mathrm{h}^{0,2} = \mathrm{h}^{2,0} = \mathrm{h}^{1,3} = \mathrm{h}^{3,1} = 6$
5. $\mathrm{h}^{1,1} = \mathrm{h}^{2,2} = 226$
6. $\mathrm{h}^{1,2} = \mathrm{h}^{2,1} = 0$

Explicit generators for the cohomology will be given.

Our result is

Theorem 1.2. *The Betti numbers*

$$\mathrm{h}^i = \dim \mathrm{H}^i(\Gamma(4), \mathbf{Q})$$

of the principal congruence subgroup of level 4 in $\mathbf{Sp}_4(\mathbf{Z})$ *are 1, 0, 118, 1112, 481 for* $i = 0, \ldots, 4$ *and 0 for all other values.*

We will give the mixed Hodge structure as well. Notice that the Euler characteristic is -512 which is the predicted value from the formula of Siegel:

$$-512 = \zeta(-1)\zeta(-3)[\Gamma(1) : \Gamma(4)] = \left(\frac{-1}{1440}\right) 737280$$

2 The building

We will first describe the (1, 1) and (2, 2) part of the cohomology. This is entirely given by algebraic cycles of a canonical sort. The description of these and their incidence relations is given by a finite geometry $\overline{\mathfrak{T}}$ called the *Tits building with scaffolding* in [12]. It is a disjoint union

$$\overline{\mathfrak{T}} \ = \ \overline{\mathfrak{P}} \cup \overline{\mathfrak{Q}}$$

To explain these, let us introduce some terminology. We say that a vector $v \in R^n$ is *primitive* if the ideal in R generated by the coordinates of v is the unit ideal. More generally, a multivector

$$h = v_1 \wedge \ldots \wedge v_s \in \bigwedge^s R^n$$

is primitive if the ideal generated by the Pluecker coordinates of it is the unit ideal in R. These concepts will only be used for R a field, $\mathbf{Z}$ or $\mathbf{Z}/n$. In those cases, primitive is equivalent to the condition that the submodule of R^n generated by h, ie. that generated by $v_1, \ldots, v_s$, is a free direct summand, and then necessarily a complement module is also free. Let us say that v divides, or is incident upon, h if we can write $\pm h = v \wedge v_2' \wedge \ldots \wedge v_s'$. This is equivalent to say that the submodule generated by v is a direct factor of the submodule generated by h.

Then

$$\overline{\mathfrak{P}} \ = \ \overline{\mathfrak{P}}_0 \cup \overline{\mathfrak{P}}_1 \cup \overline{\mathfrak{P}}_2$$

where

$\overline{\mathfrak{P}}_1 = \{$equivalence classes modulo $l \sim -l$ of primitive vectors $l \in (\mathbf{Z}/4)^4\}$

$\overline{\mathfrak{P}}_2 = \{$equivalence classes modulo $h \sim -h$ of primitive isotropic bivectors

$$h = u \wedge v \in \bigwedge^2 (\mathbf{Z}/4)^4\}$$

$\overline{\mathfrak{P}}_0 = \{$pairs (l, h) with $l \in \overline{\mathfrak{P}}_1, h \in \overline{\mathfrak{P}}_2$ and l divides $h\}$

A bivector $h = u \wedge v$ is isotropic if it is so for the standard alternating form:

$$\langle u, v\rangle \ = \ u_1v_3 + u_2v_4 - (u_3v_1 + u_4v_2) \ = \ 0$$

By a *nonsingular pair* or *anisotropic splitting* we understand a decomposition

$$(\mathbf{Z}/4)^4 \;=\; \delta_1 \oplus \delta_2$$

where each of δ_1 and δ_2 are self - dual relative to $\langle\ \rangle$. Thus, δ_1 and δ_2 are free of rank 2, and each is the orthogonal complement of the other relative to $\langle\ \rangle$:

$$\delta_1 \;=\; \delta_2^{\perp}$$

An anisotropic splitting $\Delta = \{\delta_1, \delta_2\}$ is entirely determined by a submodule which can be generated by vectors u, v such that $\langle u, v\rangle = \pm 1$. We let

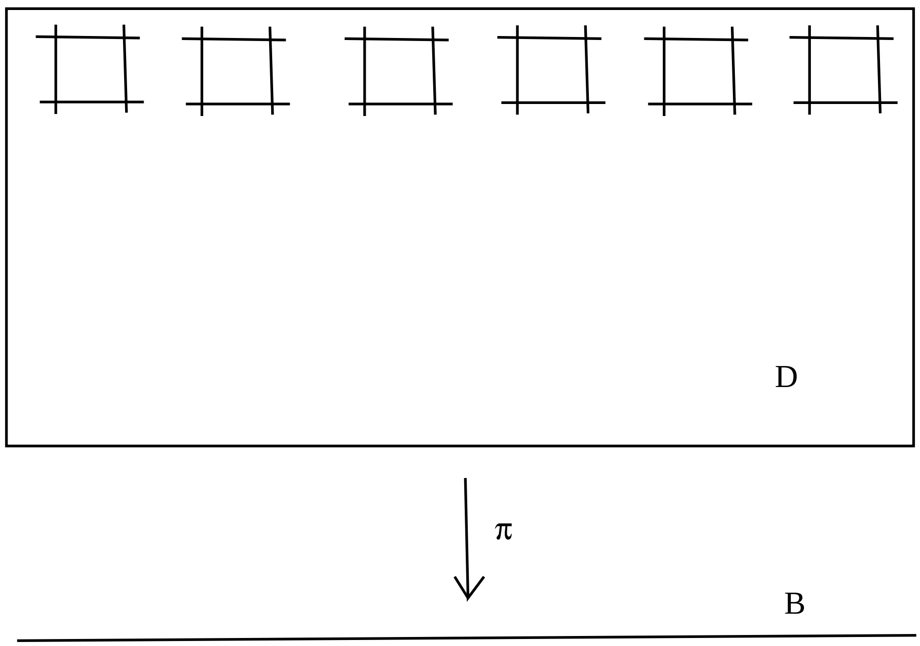
$$\overline{\mathfrak{Q}} \;=\; \{\text{ anisotropic splittings } \Delta = \{\delta_1, \delta_2\} \text{ of } (\mathbf{Z}/4)^4\}$$

We can think of the elements of $\overline{\mathfrak{P}}_i$ as the points (resp. as the isotropic lines) in the finite projective space $\mathbf{P}^3(\mathbf{Z}/4)$ for $i = 1, 2$, while those of $\overline{\mathfrak{Q}}$ are the perpendicular pairs of anisotropic (or self - dual) lines in that space. The set $\overline{\mathfrak{P}}$ is the quotient by $\Gamma(4)$ of the set of rational parabolic subgroups of $\mathbf{Sp}_4$. These correspond to the stabilizers of lines, isotropic planes, and of isotropic flags in $\mathbf{Q}^4$.

3 Cycles

3.1 The boundary

The boundary $\partial\mathcal{A}_2(4)^* = \mathcal{A}_2(4)^* - \mathcal{A}_2(4)$ is a divisor with normal crossings whose 120 components $D(l)$ are naturally indexed by $l \in \overline{\mathfrak{P}}_1$ Each $D = D(l)$ is the elliptic modular surface of level 4. There is a mapping $\pi : D \to B$ where B is the compactified modular curve of level 4. B is simply $\mathbf{P}^1$ with 6 distinguished points, the cusps. π^{-1} of a general point is a genus 1 smooth curve, and π^{-1} of a cusp is a 4 - gon of $\mathbf{P}^1$'s (this is of type I in Kodaira's classification of degenerate fibers in fibrations of elliptic curves). D is the universal family of elliptic curves with a level 4 structure. There are 16 sections of the map π corresponding to the universal points of order 4. D is a K3 surface whose Picard number is the maximum posssible (20) and the Picard group is generated by the classes of the 16 sections and the 24 cuspidal $\mathbf{P}^1$'s. We can give a basis of Pic$\otimes\mathbf{Q}$ as follows: Choose any one of the 16 sections. This section meets each of the cuspidal 4 - cycles in exactly one $\mathbf{P}^1$. For each of the 6 cusps take the 3 $\mathbf{P}^1$'s that do not meet this section. We get 20 independent divisors by taking these 18, the chosen section, and one general fiber of π. For a thorough discussion of elliptic modular surfaces, see Shioda's paper [26]. Here is a picture of a typical boundary component:



The $D(l)$ intersect in the union of subsets $C(h)$ indexed by the $h \in \overline{\mathfrak{P}}_2$. These sets are all disjoint from one another, and each one can be thought of as the space obtained by taking the 1 - skeleton of a cube and replacing the segments by $\mathbf{P}^1$'s. We have $D(l_1) \cap D(l_2) = \phi$ unless $l_1 \wedge l_2 \in \overline{\mathfrak{P}}_2$, in which case it is a cuspidal $\mathbf{P}^1$ on each of $D(l_1), D(l_2)$ and moreover every cuspidal $\mathbf{P}^1$ on each boundary component arises this way as the intersection with another boundary component. Also, $D(l_1) \cap D(l_2) \cap D(l_3)$ will be a point if and only if $l_1 \wedge l_2 = l_1 \wedge l_3 = l_2 \wedge l_3 \in \overline{\mathfrak{P}}_2$. Notation:

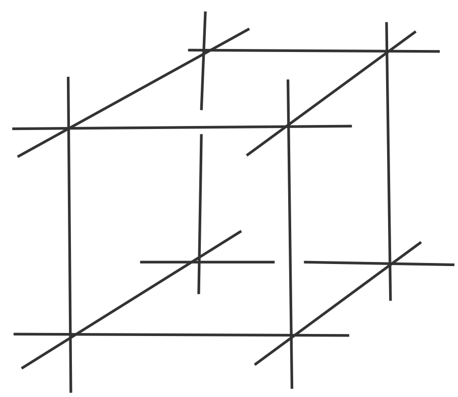
$$D(l_1,\ l_2) \ = \ D(l_1) \cap D(l_2) \ \text{ for } l_1 \wedge l_2 \in \overline{\mathfrak{P}}_2$$

These are projective lines, and there are 1440 of them.

$$D(l_1,\ l_2,\ l_3) \ = \ D(l_1) \cap D(l_2) \cap D(l_3) \ \text{ for } l_1 \wedge l_2 = l_1 \wedge l_3 = l_2 \wedge l_3 \in \overline{\mathfrak{P}}_2$$

These are points and there are 960 of them.

Here is a picture of $C(h)$:



3.2 Humbert surfaces

To each anisotropic splitting Δ there corresponds a divisor

$$H(\Delta) \;\simeq\; B \times B \;=\; \mathbf{P}^1 \times \mathbf{P}^1$$

There are 160 of these and they are all disjoint from each other. One can characterize the Humbert surfaces as follows:

$$\left(\bigcup_{\Delta} H(\Delta)\right) \cap \mathcal{A}_2(n)$$

is the zero set of the unique cusp form ψ of weight 10 for the group $\Gamma_2(1)$. This $\psi(\tau) = \theta(\tau)^2$ where $\theta(\tau)$ is the product of the 10 even theta constants with half - integer characteristics.

There are 12 distinguished curves on a Humbert surface, namely the curves

$$\mathit{cusp} \times B \text{ and } B \times \mathit{cusp}$$

These are all of the form $H(\Delta) \cap D(l)$ for the l that are on Δ, that is, on δ or $\delta^\perp$. On $D(l)$ this curve will be one of the 16 universal sections, and every such section arises this way.

The indexing for the boundary components and Humbert surfaces is equivariant for the natural actions of $\mathbf{Sp}_4(\mathbf{Z}/4)$:

$$g.D(l) \;=\; D(g.l) \text{ and } g.H(\Delta) = H(g.\Delta)$$

In particular, there is only one orbit of boundary components and one orbit of Humbert surfaces.

In [18] it is shown that:

Proposition 3.1. *The cycle classes of the 120 boundary components and the 160 Humbert surfaces generate* $\mathrm{H}^{1,1}(\mathcal{A}_2(4)^*)$.

3.3 The spectral sequence

3.3.1

We are going to compute the cohomology groups of $\Gamma_2(3)$ by using the Leray spectral sequence for the inclusion

$$j : \ \mathcal{A}_2(n) \ \hookrightarrow \ \mathcal{A}_2(n)^*$$

$$\mathrm{E}_2^{p,q} \ = \ \mathrm{H}^p(\mathcal{A}_2(n)^*, \ \mathrm{R}^q\, j_*\mathbf{Q}) \ \Longrightarrow \ \mathrm{H}^{p+q}(\mathcal{A}_2(n), \ \mathbf{Q})$$

for $n = 4$ but first let us recall some general facts for arbitrary n, due to Oda and Schwermer [21]. The corresponding sequence was analyzed by Deligne [4] for an arbitrary inclusion of the complement of a divisor with normal crossings $X \hookrightarrow \overline{X}$ on a smooth complete variety $\overline{X}$. It is a theorem of Deligne (loc. cit.) that this spectral sequence degenerates in E_3, and the resulting filtration on the abutment is, up to a renumbering, the weight filtration of the mixed Hodge structure on the latter. More precisely,

$$\mathrm{E}_3^{p,q} \ = \ \mathrm{E}_\infty^{p,q} \ = \ \mathrm{Gr}^W_{p+2q}\,\mathrm{H}^{p+q}(X, \ \mathbf{Q})$$

Recall also that $\mathrm{Gr}^W_i\,\mathrm{H}^j(X, \ \mathbf{Q}) = 0$ unless $j \le i \le 2j$ and that $\mathrm{Gr}^W_i\,\mathrm{H}^i(X, \ \mathbf{Q})$ is the image of

$$\mathrm{H}^i(\overline{X}, \ \mathbf{Q}) \ \longrightarrow \ \mathrm{H}^i(X, \ \mathbf{Q})$$

for any smooth compactification $\overline{X}$ of X.

Let

$$D^{[q]} \ = \ \text{disjoint union of all the } D(l_1, \ldots, l_q) \text{ for } l_1 < \cdots < l_q$$

where $D^{[0]} = \mathcal{A}_2(n)^*$. This is empty if $q > 3$. Then

$$\begin{aligned} \mathrm{H}^p(\mathcal{A}_2(n)^*, \ \mathrm{R}^q\, j_*\mathbf{Q}) \ &= \ \mathrm{H}^p(D^{[q]}, \ \mathbf{Q}) \\ &= \bigoplus_{I_q} \mathrm{H}^p(D(l_1, \ldots, l_q), \ \mathbf{Q}) \end{aligned}$$

where I_q is the set of $l_1 < \cdots < l_q$ in some arbitrary order of the l.

Theorem 3.2. *[21]*

$$\mathrm{H}^2(\mathcal{A}_2(n)^*,\ \mathbf{Q}) \longrightarrow \mathrm{H}^2(\mathcal{A}_2(n),\ \mathbf{Q})$$

is onto. In other words, $\mathrm{H}^2(\mathcal{A}_2(n),\ \mathbf{Q})$ *is pure of weight 2.*

This computes the second Betti number of $\mathcal{A}_2(n)$ from that of $\mathcal{A}_2(n)^*$ because of the lemma below, whose proof can be found in [11]:

Lemma 3.3.

$$\mathrm{E}_2^{0,1} = \bigoplus_l \mathrm{H}^2(D(l),\ \mathbf{Q}) \longrightarrow \mathrm{E}_2^{2,0} = \mathrm{H}^2(\mathcal{A}_2(n)^*,\ \mathbf{Q})$$

is injective. Therefore

$$\dim \mathrm{H}^2(\mathcal{A}_2(n),\ \mathbf{Q}) = \dim \mathrm{H}^2(\mathcal{A}_2(n)^*,\ \mathbf{Q}) - \#\mathfrak{P}_1(\mathbf{Z}/n)$$

Therefore, $\mathrm{H}^2(\mathcal{A}_2(4),\ \mathbf{Q})$ *is 118 dimensional and pure of weight 2 as claimed in the main theorem.*

We introduce the complexes

$$S^\bullet : \mathrm{E}_2^{0,2} \xrightarrow{d_S^1} \mathrm{E}_2^{2,1} \xrightarrow{d_S^2} \mathrm{E}_2^{4,0}$$
$$T^\bullet : \mathrm{E}_2^{0,3} \xrightarrow{d_T^1} \mathrm{E}_2^{2,2} \xrightarrow{d_T^2} \mathrm{E}_2^{4,1} \xrightarrow{d_T^3} \mathrm{E}_2^{6,0}$$

d_S^1 is injective since $\mathrm{Gr}_i^W \mathrm{H}^2 = 0$ for $i = 4$. Also, $T^\bullet$ is exact in T^2 and T^3 because $\mathrm{H}^i(\mathcal{A}_2(n),\ \mathbf{Q}) = 0$ for $i \geq 5$. We have

Theorem 3.4. *[21]*

$$\mathrm{Gr}_i^W \mathrm{H}^2(\mathcal{A}_2(n)^*,\ \mathbf{Q}) = \begin{cases} \mathrm{H}^2(\mathcal{A}_2(n)^*,\ \mathbf{Q})/\oplus \mathrm{H}^0(D(l),\ \mathbf{Q}) & \textit{for } i = 2 \\ 0 & \textit{otherwise} \end{cases}$$

$$\mathrm{Gr}_i^W \mathrm{H}^3(\mathcal{A}_2(n)^*,\ \mathbf{Q}) = \begin{cases} \mathrm{H}^3(\mathcal{A}_2(n)^*,\ \mathbf{Q})/\oplus \mathrm{H}^1(D(l),\ \mathbf{Q}) & \textit{for } i = 3 \\ \mathrm{H}^1(S) & \textit{for } i = 4 \\ 0 & \textit{for } i = 5 \\ \mathrm{H}^0(T) & \textit{for } i = 6 \\ 0 & \textit{otherwise} \end{cases}$$

$$\mathrm{Gr}_i^W \mathrm{H}^4(\mathcal{A}_2(n)^*,\ \mathbf{Q}) = \begin{cases} \mathrm{H}^2(S) & \textit{for } i = 4 \\ \oplus \mathrm{H}^3(D(l),\ \mathbf{Q}) & \textit{for } i = 5 \\ \mathrm{H}^1(T) & \textit{for } i = 6 \\ 0 & \textit{otherwise} \end{cases}$$

Each differential

$$d : \bigoplus \mathrm{H}^p(D(l_1, \ldots, l_q),\ \mathbf{Q}) \longrightarrow \bigoplus \mathrm{H}^p(D(l_1, \ldots, l_{q-1}),\ \mathbf{Q})$$

is of the form $d = \sum (-1)^i d_{i*}$ where d_{i*} is the Gysin homomorphism associated to the inclusion

$$d_i : D(l_1, \ldots, l_q) \longrightarrow D(l_1, \ldots, \widehat{l_i}, \ldots, l_{q-1})$$

These Gysin maps are easy to understand when applied to cohomology classes of algebraic cycles. If x is such a class, associated to a codimension r cycle on $D(l_1, \ldots, l_q)$, then $d_{i*}(x)$ is the class of that same cycle, but now of codimension $r+1$ on

$$D(l_1, \ldots, \widehat{l_i}, \ldots, l_q)$$

The above theorem shows that to compute the Betti numbers of $\Gamma_2(n)$, one needs to compute the cohomology of the Igusa compactification and the ranks of the operators d_S^i, d_T^i. The ranks of d_S^1, d_T^2, d_T^3 are known, as we have already mentioned. We can determine the rank of d_T^1:

Lemma 3.5. $\dim \mathrm{Ker}(d_T^1) \ = \ \#\mathfrak{P}_2(\mathbf{Z}/n)$

The proof is given in [11]. In analyzing the differential $d = d_S^2$, first observe that these Gysin homomorphisms are morphisms of type (1, 1) for the Hodge decompositions. We can therefore decompose

$$d \ = \ d_{\mathrm{alg}} + d_{\mathrm{tr}} + \overline{d}_{\mathrm{tr}} \qquad \text{where} \qquad \overline{d}_{\mathrm{tr}} \text{ is the complex conjugate of } d_{\mathrm{tr}}$$

$$d_{\mathrm{alg}} : \bigoplus_l \mathrm{H}^{1,1}(D(l)) \longrightarrow \mathrm{H}^{2,2}(\mathcal{A}_2(n)^*)$$

$$d_{\mathrm{tr}} : \bigoplus_l \mathrm{H}^{2,0}(D(l)) \longrightarrow \mathrm{H}^{3,1}(\mathcal{A}_2(n)^*)$$

Proposition 3.6. *Suppose that* $\mathrm{H}^{1,1}(\mathcal{A}_2(n)^*)$ *is generated by the cycle classes of the* $D(l)$ *and the* $H(\Delta)$*. Then* d_{alg} *is surjective.*

See [11]. From this point on $n = 4$, unless otherwise stated. Because of proposition 3.1 we have

Corollary 3.7. *The rank of* $(d_S^2)_{alg}$ *is 226.*

Proposition 3.8. *d_{tr} and $\overline{d}_{tr}$ are surjective.*

Proof. Enough to do this for $\overline{d}_{\rm tr}$. Let

$$W \subset \mathrm{H}^{1,3}(\mathcal{A}_2(4)^*)$$

be the image of $\overline{d}_{\rm tr}$. Since cup - product

$$\mathrm{H}^{2,0}(\mathcal{A}_2(4)^*) \otimes \mathrm{H}^{1,3}(\mathcal{A}_2(4)^*) \;\longrightarrow\; \mathrm{H}^{3,3}(\mathcal{A}_2(4)^*) = \mathbf{C}$$

is a perfect pairing, it will be enough to show that the orthogonal

$$W^{\perp} \subset \mathrm{H}^{2,0}(\mathcal{A}_2(4)^*)$$

is zero. We can write $\overline{d}_{\rm tr} = \sum d_{l*}$, where

$$d_{l*} : \mathrm{H}^{0,2}(D(l)) \;\longrightarrow\; \mathrm{H}^{1,3}(\mathcal{A}_2(4)^*)$$

Thus

$$W^{\perp} \;=\; \{\omega \in \mathrm{H}^{2,0}(\mathcal{A}_2(4)^*) : d_{l*}(\theta)\cdot\omega = 0 \text{ for all } l \text{ and all } \theta \in \mathrm{H}^{0,2}(D(l))\}$$

But $d_{l*}(\theta)\cdot\omega = d_{l*}(\theta\cdot d_l^*(\omega))$ by the projection formula. The d_{l*} in the right - hand side of the above equation is an isomorphism

$$\mathrm{H}^{2,2}(D(l)) \;\simeq\; \mathrm{H}^{3,3}(\mathcal{A}_2(4)^*)$$

so the condition $d_{l*}(\theta)\cdot\omega = 0$ is equivalent with $\theta\cdot d_l^*(\omega) = 0$. For a fixed l this holds for all θ if and only if $d_l^*(\omega) = 0$. Hence,

$$W^{\perp} \;=\; \{\omega \in \mathrm{H}^{2,0}(\mathcal{A}_2(4)^*) : d_l^*(\omega) = 0 \text{ for all } l\}$$

$W^{\perp}$ is a $G = \mathbf{PSp}(4, \mathbf{Z}/4)$ - submodule of the $(2,0)$ - forms on $\mathcal{A}_2(4)^*$. It will be shown in lemma 3.9 that the action is irreducible on the latter space. Therefore, if there were a nonzero ω in $W^{\perp}$, it would follow that $d_l^*(\omega) = 0$ for every l and every $(2,0)$ - form ω. This is not so. Weissauer has provided an explicit basis for these forms, and in particular one of them is

$$\omega \;=\; \sum_{x,y\in\mathbf{Z}[i]} ((x+1/2)^2 d\tau_1\wedge d\tau_2 + (x+1/2)y\, d\tau_1\wedge d\tau_3 + y^2\, d\tau_2\wedge d\tau_3)$$
$$\times \exp(2\pi i(x+1/2,y)\,\tau\,{}^t\overline{(x+1/2,y)})$$

where

$$\tau \;=\; \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \quad \text{is a point in } \mathfrak{S}_2$$

We claim that this has a nonzero restriction to the boundary component D where $\tau_3 = i\infty$. Let $q_3 = \exp(2\pi i \tau_3/4)$. Then it is known that $(\tau_1.\tau_2, q_3)$ is a system of local coordinates in a neighborhood of D, that divisor being defined by $q_3 = 0$. We have

$$\omega \;=\; \sum_{x,y\in\mathbf{Z}[i]} ((x+1/2)^2 d\tau_1 \wedge d\tau_2 + (2/\pi i)(x+1/2)y\, d\tau_1 \wedge dq_3/q_3$$
$$+\, (2/\pi i)y^2\, d\tau_2 \wedge dq_3/q_3) \times \exp(2\pi i(|x+1/2|^2\tau_1 + 2\,\mathrm{Re}(y\overline{(x+1/2)})\tau_2))q_3^{4|y|^2}$$

Note that the apparent poles at $q_3 = 0$ are spurious. At any rate, letting $q_3 \to 0$ we see that every term with $y \neq 0$ goes to 0 and we are left with

$$\left(\sum_{x\in\mathbf{Z}[i]} (x+1/2)^2 \exp(2\pi i(|x+1/2|^2\tau_1)\right) d\tau_1 \wedge d\tau_2$$

This is nonzero. In fact, the expression in parenthesis is a cusp form of weight 3 for the group $\Gamma_1(4)$, as it must be since Shioda has shown [33] that there is a natural identification as above between the space of weight 3 cusp forms and the $(2,0)$ forms on any elliptic modular surface. Let $q = \exp(2\pi i \tau_1/4)$ and multiply the above sum by 4. We get the expansion

$$\sum_{x\in\mathbf{Z}[i]} (2x+1)^2 q^{|2x+1|^2}$$

This is equal to

$$2\,\Delta(\tau_1)^{1/4} \;=\; 2\,q\prod_{n\geq 1}(1-q^{4n})^6$$

as can be seen by computing a few terms. This is the unique (up to scalar multiples) cusp form of weight 3 for $\Gamma_1(4)$. The expression of this as a sum reveals it as a cusp form attached to a Hecke character (Grössencharakter) of conductor 2 for the field $\mathbf{Q}(i)$. See [25]. $\square$

There is an exact sequence

$$0 \;\longrightarrow\; \mathfrak{psp}(4,\mathbf{Z}/2) \;\longrightarrow\; G \;\longrightarrow\; \mathbf{PSp}(4,\mathbf{Z}/2) \longrightarrow\; 0$$

the subgroup on the left being a vector space of dimension 9 over the field with 2 elements. $\mathbf{PSp}(4, \mathbf{Z}/2)$ operates by conjugation on the characters of $\mathfrak{psp}(4, \mathbf{Z}/2)$, and there are 10 orbits for this action. Any representation of G will decompose into a direct sum of various character eigenspaces for the action of this subgroup. In [18] it is shown that

$$\mathrm{H}^{2,0}(\mathcal{A}_2(4)^*) \; = \; \bigoplus \mathrm{H}^{2,0}(\mathcal{A}_2(4)^*)_\lambda$$

where the direct sum is over the 6 characters λ belonging to the orbit denoted IIB in that work, and each of the eigenspaces above is 1 - dimensional. The result needed in the previous proposition follows from the (no doubt well - known):

Lemma 3.9. *Let*

$$0 \longrightarrow A \longrightarrow G \longrightarrow H \longrightarrow 0$$

be an exact sequence of finite groups with A abelian. Let

$$V \; = \; \bigoplus V_\lambda$$

be a representation of G in a field of characteristic 0 decomposed into A - character eigenspaces. Suppose that the λ appearing constitute one orbit for the natural H - action on the characters of A, and that each V_λ is one dimensional. Then V is irreducible.

Proof. We will show that the G - module generated by any nonzero vector is all of V. Let $v = (v_1, \dots, v_r)$ written in the basis of λ - eigenvectors. If $r = 1$ the result is clear. Without loss of generality we can assume that $v_1 \neq 0$. There is an $a \in A$ such that $\lambda_1(a) \neq \lambda_2(a)$. Then

$$\lambda_2(a)(v_1, v_2, \dots, v_r) - a \cdot (v_1, v_2, \dots, v_r)$$

is in the G - space spanned by v and has first coordinate nonzero, but second coordinate 0. Proceeding in this way we get a vector of the form $(*, 0, \dots, 0)$, in other words, a λ_1 - eigenvector, in the G- span of v. Since G operates transitively on the characters appearing in V we get λ - eigenvectors for every λ, hence a basis. ☐

We have computed all the differentials in the spectral sequence. Here is the main result:

Theorem 3.10. *Let* $\mathrm{h}^{p,q}(m)$ *denote the dimension of the* $(p.q)$ *- part of the Hodge structure of pure weight*

$$\mathrm{Gr}^W_{p+q}(\mathrm{H}^m(\mathcal{A}_2(4),\ \mathbf{C}))$$

The nonzero ones are

1. $\mathrm{h}^{0,0}(0) = 1$
2. $\mathrm{h}^{1,1}(2) = 106,\ \mathrm{h}^{2,0}(2) = \mathrm{h}^{0,2}(2) = 6$
3. $\mathrm{h}^{3,0}(3) = \mathrm{h}^{3,0}(3) = 15,\ \mathrm{h}^{2,2}(3) = 734,\ \mathrm{h}^{3,1}(3) = \mathrm{h}^{1,3}(3) = 114,$
   $\mathrm{h}^{3,3}(3) = 120$
4. $\mathrm{h}^{3,3}(4) = 481$

Theorem 3.11. *Let* $\mathrm{IH}^m(\mathcal{A}_2(4)^{sa}, \mathbf{Q})$ *denote the intersection cohomology with middle perversity of the Satake compactification. The dimensions are*

$$1, 0, 118, 30, 118, 0, 1$$

for $m = 0, \dots, 6$ *and 0 for all others.*

Proof. After tensoring with $\mathbf{C}$, the intersection cohomology becomes isomorphic with the L^2 - cohomology

$$\mathrm{H}^m_{(2)}(\mathcal{A}_2(4), \mathbf{C})$$

by the solution to Zucker's conjecture given by Looijenga [19], Saper - Stern [24], Ohsawa [22]. Oda and Schwermer [21] showed that the natural map

$$\mathrm{H}^2_{(2)}(\mathcal{A}_2(n), \mathbf{C}) \ \longrightarrow\ \mathrm{H}^2(\mathcal{A}_2(n), \mathbf{C})$$

is surjective and Weissauer [30] proved that it is injective, so it is an isomorphism, for any n. This proves the claims about m = 2. We get the result for m = 4 by Poincaré duality which shows a canonical isomorphism:

$$\mathrm{IH}^4(\mathcal{A}_2(4)^{\mathrm{sa}}) \ =\ \mathrm{Gr}^W_4\,\mathrm{H}^4_c(\mathcal{A}_2(4))$$

To get the assertions about m = 1, 5, observe that the decomposition theorem [5] shows that the intersection cohomology of the Satake compactification is a direct summand of the cohomology of a resolution of singularities of it, but we have seen that these groups are all 0. To get the result about m = 3, use the inequality of Durfee [6, Prop. 5]

$$\dim \mathrm{Gr}^W_k\,\mathrm{H}^k(\mathcal{A}_2(4)) \ \le\ \dim \mathrm{IH}^k(\mathcal{A}_2(4)^{\mathrm{sa}}) \ \le\ \dim \mathrm{H}^k(\mathcal{A}_2(4)^*)$$

Both lower and upper bounds are 30 dimensional. This shows more precisely an isomorphism of Hodge structures

$$\mathrm{IH}^3(\mathcal{A}_2(4)^{\mathrm{sa}}) \;=\; \mathrm{H}^3(\mathcal{A}_2(4)^*)$$

Alternatively, we can derive this theorem from the general results of Durfee, which depend on the theory of M. Saito [23]. □

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