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SYMMETRIC SUBGROUP ACTIONS ON ISOTROPIC GRASSMANNIANS

HONGYU HE AND HUAJUN HUANG

ABSTRACT. Let G be a classical group preserving a sesquilinear form on a vector space V over $\mathbb{R}$ or $\mathbb{C}$. Let $\mathrm{Gr}_G(r)$ be the Grassmannian of isotropic r-dimensional subspaces. Let $H = (G_1, G_2)$ be a symmetric subgroup of G. In this paper, we give a parametrization of H-orbits on $\mathrm{Gr}_G(r)$ in terms of dimensions of various subspaces. The main result of this paper is the determination of the H homogeneous structure and the dimension of each orbit. Consequently, we find all the open orbits. We also treat H-orbits of $\mathrm{Gr}_G(r)$ for symplectic and orthogonal groups over an algebraic closed field with characteristic not equal to 2.

1. Introduction

The symmetric subgroup orbits in flag manifolds have been intensively studied in the past. Their parametrization, in the most general form, is due to Matsuki [10, 11, 12, 13] and Springer [16]. There are finitely many such orbits. In addition, there is also a natural topological ordering among the symmetric subgroup orbits, namely, if an orbit $\mathcal{O}'$ is contained in the Zariski closure of another orbit $\mathcal{O}$, $\mathcal{O}'$ is said to be smaller than $\mathcal{O}$. This ordering defines a partial ordering, often called the Bruhat ordering. The Bruhat ordering can be described purely algebraically in terms of the Matsuki-Springer parameter [11, 14, 15, 6].

In this paper, we are interested in symmetric subgroup action on the Grassmannian of isotropic subspaces. Consider the following symmetric pairs.

TABLE 1. Symmetric Pairs (G, H) and Representation Spaces V

G	H	V	Conditions
$\mathrm{O}(p,q)$	$\mathrm{O}(p_1,q_1)\times \mathrm{O}(p-p_1,q-q_1)$	$\mathbb{R}^{p_1+q_1}\oplus\mathbb{R}^{p-p_1+q-q_1}$	$0<p_1<p,\ 0<q_1<q$
$\mathrm{U}(p,q)$	$\mathrm{U}(p_1,q_1)\times \mathrm{U}(p-p_1,q-q_1)$	$\mathbb{C}^{p_1+q_1}\oplus\mathbb{C}^{p-p_1+q-q_1}$	$0<p_1<p,\ 0<q_1<q$
$\mathrm{O}_n(\mathbf{k})$	$\mathrm{O}_m(\mathbf{k})\times\mathrm{O}_{n-m}(\mathbf{k})$	$\mathbf{k}^m\oplus\mathbf{k}^{n-m}$	$0<m<n$
$\mathrm{Sp}_{2n}(\mathbf{k})$	$\mathrm{Sp}_{2m}(\mathbf{k})\times\mathrm{Sp}_{2n-2m}(\mathbf{k})$	$\mathbf{k}^{2m}\oplus\mathbf{k}^{2n-2m}$	$0<2m<2n$

Note: $\mathbf{k}$ is an algebraic closed field and $\mathrm{char}(\mathbf{k})\neq 2$.

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In Table 1, G is a classical group that preserves a standard sesquilinear form on the standard representation space V, and H is a symmetric subgroup of G that stabilizes two subspaces U and W such that $V = U \oplus W$. Let $\mathrm{Gr}_G(r)$ be the Grassmannian of isotropic subspaces of dimension r. Let P be a maximal parabolic subgroup such that quotient $G/P \cong \mathrm{Gr}_G(r)$.

The first result of this paper is a parametrization of the cosets $H\backslash\mathrm{Gr}_G(r)$ by a finite convex subset of a lattice.

Theorem 1.1. *Let* $\mathbf{k}$ *be an algebraically closed field with* $\mathrm{char}(\mathbf{k}) \neq 2$. *Let* (G, H, V) *be given as in Table 1. If* G *is* $\mathrm{O}_n(\mathbf{k})$ *or* $\mathrm{Sp}_{2n}(\mathbf{k})$, *then* H*-orbits on* $\mathrm{Gr}_G(r)$ *can be parametrized by an integral 4-tuple* (r_U, r_W, a, b)*; If* $G = \mathrm{O}(p, q)$ *or* $G = \mathrm{U}(p, q)$, *then* H*-orbits on* $\mathrm{Gr}_G(r)$ *can be parametrized by an integral 5-tuple* (r_U, r_W, a, a_U, a_W).

See Theorems 2.1, 5.1, 6.1 and 7.1 for the definitions of parameters r_U, r_W, a, b, a_U, a_W and the precise statements. The results for $(H \cap G_0)$-orbit for $\mathrm{O}(p, q)$ are given in Section 4.

Our view point is purely algebraic. We derive our theorem by analyzing the simultaneous isometry of a set of subspaces as presented in [7, Theorem 5.3]. It is unclear how our parametrization should be identified with the Matsuki-Springer parametrization [1, 10, 11, 16]. We shall point out that the symmetric subgroup orbits we treat in this paper are quite special. All the parameters can be directly determined by any isotropic subspace in the orbit. They sit in a positive lattice. We expect that the Bruhat ordering coincides with a natual ordering on the lattice.

Another focus of this paper is to study how the group H acts on each orbit. Clearly each orbit can be written as H/H_S where H_S is the stabilizer of an isotropic subspace S. The second result of this paper, is a determination of the stabilizer H_S. In the most general form, H_S is contained in a product of two parabolic subgroups $P_U(r_U, r_U + a)$ of $H|_U$ and $P_W(r_W, r_W + a)$ of $H|_W$ respectively.

Theorem 1.2. *Let* G, H *be as in Table 1. Let* S *be an isotropic subspace in the* H*-orbit* $\mathcal{O}_{G,H}(r_U, r_W, a, *)$. *Then* $H_S \subseteq P_U(r_U, r_U + a) \times P_W(r_W, r_W + a)$. *Moreover,* $(g_1, g_2) \in H_S$ *if and only if* g_1 *and* g_2 *satisfy certain matching conditions. The dimension of* $\mathcal{O}_{G,H}(r_U, r_W, a, *)$ *is a quadratic function on the 4-tuple or 5-tuple. The open orbits are determined in Theorems 3.3, 5.4, 6.3, 7.5.*

See Theorems 3.1, 5.2, 6.2, 7.2 for the precise description of the stabilizer H_S.

Let $\mathcal{F}_G$ be the complete flag variety of G, and $\mathcal{F}'_G$ any partial flag variety of G. In [11], Matsuki defined the canonical surjection

$$f : H\backslash\mathcal{F}_G \to H\backslash\mathcal{F}'_G,$$

then determined the inverse image $f^{-1}(\mathcal{O})$ for any H-orbit $\mathcal{O}$ in $\mathcal{F}'_G$. For (G, H) given in Table 1, there exists a similar canonical surjection from $H\backslash\mathcal{F}'_G$ to $H\backslash\mathrm{Gr}_G(r)$ for every $r \in \mathrm{DIM}(\mathcal{F}'_G)$, where $\mathrm{DIM}(\mathcal{F}'_G)$ is the dimension set of an isotropic flag in $\mathcal{F}'_G$. Clearly, the following canonical map π is injective:

$$\pi : \mathcal{F}'_G \to \prod_{r\in\mathrm{DIM}(\mathcal{F}'_G)} \mathrm{Gr}_G(r). \tag{1}$$

We define a canonical map

$$\widetilde{f} : H\backslash\mathcal{F}'_G \to \prod_{r\in\mathrm{DIM}(\mathcal{F}'_G)} H\backslash\mathrm{Gr}_G(r). \tag{2}$$

We show in Example 6.6 that $\widetilde{f}$ is not necessarily injective. So the H-orbit of a flag in $\mathcal{F}'_G$ may not be uniquely determined by the H-orbits of subspaces in this flag. Example 6.6 also indicates that $\widetilde{f}$ is not necessarily surjective.

We shall elaborate a little bit on the motivation of this paper. Recall that functions on isotropic Grassmannian $\mathrm{Gr}_G(r)$ can be used to define certain degenerate principal series $I_P(v)$. The representation $I_P(v)$ is one of the most intensively studied series of representations. In case G/P is the Lagrangian Grassmannian and G the symplectic group, a preliminary investigation by the first author, gives a branching law for the unitary $I_P(v)|_H$ ([4, 5]). This branching law is multiplicity free and yields a Howe type L^2-correspondence ([8], [9]) between unitary representations of G_1 and unitary representations of G_2 . So the remaining question is to see if the degenerate principal series in other cases will decompose in a similar fashion when restricted to H. A first step is to understand how H acts on $\mathrm{Gr}_G(r)$, in particular how H acts on the open orbits in $\mathrm{Gr}_G(r)$. This is what is done in this paper. We hope to discuss the branching law of the degenerate principal series with respect to H in a future paper.

2. Real Orthogonal Case: Orbits on Isotropic Grassmannians

In this section, suppose that

$$G := \mathrm{O}(p, q), \qquad H := \mathrm{O}(p_1, q_1) \times \mathrm{O}(p - p_1, q - q_1). \tag{3}$$

Here G is the isometry group of $V := \mathbb{R}^{p+q}$ with respect to a nonsingular symmetric bilinear form $(,)$, and $H \subseteq G$ is the stabilizer of two subspaces U and W of V, where

$$V = U \oplus W, \qquad U \perp W, \qquad H|_U \simeq \mathrm{O}(p_1, q_1), \qquad H|_W \simeq \mathrm{O}(p - p_1, q - q_1). \tag{4}$$

We parametrize the H-orbits in the isotropic Grassmannian $\mathrm{Gr}_G(r)$ of r-dimensional isotropic subspaces of V.

Given any subspace $S \subseteq V$, let $\mathbf{P}_U S$ (resp. $\mathbf{P}_W S$) be the *projection* of S onto the U-component (resp. the W-component) with respect to $V = U \oplus W$. So

$$\mathbf{P}_U S := (S + W) \cap U, \qquad \mathbf{P}_W S := (S + U) \cap W. \tag{5}$$

The *radical* of S with respect to the bilinear form $(\,,\,)$ is

$$\operatorname{Rad}(S) := S \cap S^{\perp}. \tag{6}$$

S is said to be *positive definite* if $(\mathbf{v}, \mathbf{v}) > 0$ (resp. *negative definite* if $(\mathbf{v}, \mathbf{v}) < 0$) for every nonzero vector $\mathbf{v} \in S$.

Recall that the G-orbit of S can be parametrized by a 3-tuple (s, s_+, s_-), where

$$s := \dim \operatorname{Rad}(S), \tag{7a}$$

$$s_+ := \text{dimension of a maximal positive definite subspace of } S, \tag{7b}$$

$$s_- := \text{dimension of a maximal negative definite subspace of } S. \tag{7c}$$

Moreover $s + s_+ + s_- = \dim S$. We say that S is *of isometry type* (s, s_+, s_-).

Now suppose that S is an r-dimensional *isotropic* subspace of V. Let

$$\mathbb{N}_0 := \{0, 1, 2, 3, \cdots\}.$$

The following theorem parametrizes the H-orbit of S.

Theorem 2.1. *Let S be an isotropic subspace of $V = \mathbb{R}^{p+q} = U \oplus W$. Define*

$$r_U := \dim(S \cap U), \tag{8a}$$

$$r_W := \dim(S \cap W), \tag{8b}$$

$$\begin{aligned} & := \dim(\operatorname{Rad}(\mathbf{P}_U S)) - r_U \\ a & = \dim(\operatorname{Rad}(\mathbf{P}_W S)) - r_W, \end{aligned} \tag{8c}$$

$$\begin{aligned} a_U & := \textit{dimension of a maximal positive definite subspace of } \mathbf{P}_U S \\ & = \textit{dimension of a maximal negative definite subspace of } \mathbf{P}_W S, \end{aligned} \tag{8d}$$

$$\begin{aligned} a_W & := \textit{dimension of a maximal negative definite subspace of } \mathbf{P}_U S \\ & = \textit{dimension of a maximal positive definite subspace of } \mathbf{P}_W S. \end{aligned} \tag{8e}$$

Then the 5-tuple $(r_U, r_W, a, a_U, a_W) \in {\mathbb{N}_0}^5$ is H-invariant, and it uniquely determines the H-orbit of S. The 5-tuple (r_U, r_W, a, a_U, a_W) satisfies the following conditions:

$$(r_U, r_W, a, a_U, a_W) \in {\mathbb{N}_0}^5, \tag{9a}$$

$$r_U + a + a_U \le p_1, \tag{9b}$$

$$r_U + a + a_W \le q_1, \tag{9c}$$

$$r_W + a + a_W \le p - p_1, \tag{9d}$$

$$r_W + a + a_U \le q - q_1. \tag{9e}$$

Theorem 2.1 shows that (r_U, r_W, a, a_U, a_W) is an integer point in a convex set of $\mathbb{R}^5$. We denote the H-orbit of S by $\mathcal{O}_{G,H}(r_U, r_W, a, a_U, a_W)$.

Remark 2.2. *Obviously,*

$$r = \dim S = r_U + r_W + a + a_U + a_W, \tag{10a}$$
$$\dim \mathbf{P}_U S = r_U + a + a_U + a_W, \tag{10b}$$
$$\dim \mathbf{P}_W S = r_W + a + a_U + a_W. \tag{10c}$$

Proof of Theorem 2.1. First we prove that (r_U, r_W, a, a_U, a_W) is well-defined by (8). The subspace S can be expressed as

$$S = (S \cap U) \oplus (S \cap W) \oplus \bigoplus_{i=1}^{k} \mathbb{R}(\mathbf{u}_i + \mathbf{w}_i), \tag{11}$$

where $\mathbf{u}_i \in U$ and $\mathbf{w}_i \in W$ for $i = 1, 2, \cdots, k$, and

$$k := \dim S - \dim(S \cap U) - \dim(S \cap W) = a + a_U + a_W.$$

Then

$$\mathbf{P}_U S = (S \cap U) \oplus \bigoplus_{i=1}^{k} \mathbb{R}\mathbf{u}_i, \qquad \mathbf{P}_W S = (S \cap W) \oplus \bigoplus_{i=1}^{k} \mathbb{R}\mathbf{w}_i.$$

Since S is isotropic and $U \perp W$, we have

$$S \cap U \subseteq \mathrm{Rad}(\mathbf{P}_U S), \qquad S \cap W \subseteq \mathrm{Rad}(\mathbf{P}_W S).$$

Select the vectors $\mathbf{u}_i + \mathbf{w}_i$ appropriately so that the Gram matrix of $\{\mathbf{u}_1, \cdots, \mathbf{u}_k\}$ is

$$\big[\,(\mathbf{u}_i, \mathbf{u}_j)\,\big]_{k\times k} = 0_{a\times a} \oplus I_{a_U} \oplus (-I_{a_W}). \tag{12}$$

Then $(\mathbf{u}_i + \mathbf{w}_i, \mathbf{u}_j + \mathbf{w}_j) = (\mathbf{u}_i, \mathbf{u}_j) + (\mathbf{w}_i, \mathbf{w}_j) = 0$ implies that the Gram matrix

$$\big[\,(\mathbf{w}_i, \mathbf{w}_j)\,\big]_{k\times k} = -\big[\,(\mathbf{u}_i, \mathbf{u}_j)\,\big]_{k\times k} = 0_{a\times a} \oplus (-I_{a_U}) \oplus I_{a_W}. \tag{13}$$

In particular, the dimension of a maximal positive definite subspace (resp. maximal negative definite subspace) of $\mathbf{P}_U S$ equals the dimension of a maximal negative definite subspace (resp. maximal positive definite subspace) of $\mathbf{P}_W S$, and

$$\dim(\mathrm{Rad}(\mathbf{P}_U S)) - \dim(S \cap U) = \dim(\mathrm{Rad}(\mathbf{P}_W S)) - \dim(S \cap W).$$

So (r_U, r_W, a, a_U, a_W) is well-defined by (8).

Second, we show that (r_U, r_W, a, a_U, a_W) defined by (8) meets conditions (9). It is obviously H-invariant and in ${\mathbb{N}_0}^5$. By (12), the subspace S_1 of U:

$$S_1 := (S \cap U) \oplus \bigoplus_{i=1}^{a+a_U} \mathbb{R}\mathbf{u}_i$$

satisfies that $(\mathbf{v}, \mathbf{v}) \geq 0$ for every nonzero $\mathbf{v} \in S_1$. Let U_1 be a maximal negative definite subspace of U. Then $\dim U_1 = q_1$, $S_1 \cap U_1 = \{\mathbf{0}\}$ and $S_1 + U_1 \subseteq U$. So

$$\dim(S_1 + U_1) = \dim S_1 + \dim U_1 = r_U + a + a_U + q_1 \leq \dim U = p_1 + q_1.$$

Thus (9b) holds. Similarly, (9c), (9d), and (9e) hold.

Next we show that every isotropic subspace S' corresponding to (r_U, r_W, a, a_U, a_W) is in the H-orbit of S. Parallel to (11), (12), (13), S' has a decomposition:

$$S' = (S' \cap U) \oplus (S' \cap W) \oplus \bigoplus_{i=1}^{k} \mathbb{R}(\mathbf{u}'_i + \mathbf{w}'_i),$$

where

$$\big[\,(\mathbf{u}'_i, \mathbf{u}'_j)\,\big]_{k\times k} = 0_{a\times a} \oplus I_{a_U} \oplus (-I_{a_W}), \quad \big[\,(\mathbf{w}'_i, \mathbf{w}'_j)\,\big]_{k\times k} = 0_{a\times a} \oplus (-I_{a_U}) \oplus I_{a_W}.$$

Let $\phi : \mathbf{P}_U S \to \mathbf{P}_U S'$ be a linear bijection such that

$$\phi(S \cap U) = S' \cap U, \qquad \phi(\mathbf{u}_i) = \mathbf{u}'_i \quad \text{for} \quad i = 1, 2, \cdots, k.$$

Then ϕ is an isometry. By Witt's Theorem [17], ϕ can be extended to an isometry $g_1 \in \mathrm{O}(U)$, the orthogonal group of U. Likewise, there is an isometry $g_2 \in \mathrm{O}(W)$ such that

$$g_2(S \cap W) = S' \cap W, \qquad g_2(\mathbf{w}_i) = \mathbf{w}'_i \quad \text{for} \quad i = 1, 2, \cdots, k.$$

Then $g_1 \times g_2 \in H$ and it sends S to S'. Hence S and S' are in the same H-orbit.

Finally, given any 5-tuple (r_U, r_W, a, a_U, a_W) that satisfies conditions (9), we show that there exists an H-orbit corresponding to (r_U, r_W, a, a_U, a_W). In this situation S is not given. Let $\{\mathbf{u}_1^+, \cdots, \mathbf{u}_{p_1}^+, \mathbf{u}_1^-, \cdots, \mathbf{u}_{q_1}^-\}$ and $\{\mathbf{w}_1^+, \cdots, \mathbf{w}_{p-p_1}^+, \mathbf{w}_1^-, \cdots, \mathbf{w}_{q-q_1}^-\}$ be fixed orthogonal bases of U and W respectively such that for all applicable indices,

$$(\mathbf{u}_i^+, \mathbf{u}_i^+) = 1, \qquad (\mathbf{u}_j^-, \mathbf{u}_j^-) = -1, \qquad (\mathbf{w}_t^+, \mathbf{w}_t^+) = 1, \qquad (\mathbf{w}_\ell^-, \mathbf{w}_\ell^-) = -1.$$

Let S be the subspace spanned by the basis

$$\begin{aligned}(14) \quad & \{\mathbf{u}_i^+ + \mathbf{u}_i^-\}_{i=1}^{r_U} \cup \{\mathbf{w}_j^+ + \mathbf{w}_j^-\}_{j=1}^{r_W} \cup \{\mathbf{u}_{r_U+c}^+ + \mathbf{w}_{r_W+c}^-\}_{c=1}^{a_U} \cup \{\mathbf{u}_{r_U+d}^- + \mathbf{w}_{r_W+d}^+\}_{d=1}^{a_W} \\ & \cup \{\mathbf{u}_{r_U+a_U+\ell}^+ + \mathbf{u}_{r_U+a_W+\ell}^- + \mathbf{w}_{r_W+a_W+\ell}^+ + \mathbf{w}_{r_W+a_U+\ell}^-\}_{\ell=1}^{a}.\end{aligned}$$

Then S is a canonical isotropic subspace of V and the H-orbit of S is parametrized by (r_U, r_W, a, a_U, a_W). We complete the proof. □

3. Real Orthogonal Case: Homogeneous Structures of Orbits

We continue the discussion on $(G, H) = (\mathrm{O}(p, q),\ \mathrm{O}(p_1, q_1) \times \mathrm{O}(p - p_1, q - q_1))$. First we parametrize the stabilizer H_S of any isotropic subspace S of V. Then we determine the open H-orbits in $\mathrm{Gr}_G(r)$. From (9) there are only finitely many H-orbits in $\mathrm{Gr}_G(r)$, and there exists at least one open H-orbit in $\mathrm{Gr}_G(r)$.

Let us parametrize H_S for $S \in \mathcal{O}_{G,H}(r_U, r_W, a, a_U, a_W)$. Let $P_U(r_U, r_U + a)$ be the parabolic subgroup of $H|_U \cong O(p_1, q_1)$ preserving the partial flag

$$U \cap S \subseteq \mathrm{Rad}(\mathbf{P}_U S) \subseteq [\mathrm{Rad}(\mathbf{P}_U S)]^\perp \subseteq (U \cap S)^\perp \quad \text{(all are restricted in } U).$$

Let $P_W(r_W, r_W + a)$ be the parabolic subgroup of $H|_W$ preserving the partial flag:

$$W \cap S \subseteq \mathrm{Rad}(\mathbf{P}_W S) \subseteq [\mathrm{Rad}(\mathbf{P}_W S)]^\perp \subseteq (W \cap S)^\perp \quad \text{(all are restricted in } W).$$

Then $H_S \subseteq P_U(r_U, r_U + a) \times P_W(r_W, r_W + a)$. The Levi factor of $P_U(r_U, r_U + a)$ is $\mathrm{GL}(r_U)\mathrm{GL}(a)\mathrm{O}(p_1 - r_U - a, q_1 - r_U - a)$ and the Levi factor of $P_W(r_W, r_W + a)$ is $\mathrm{GL}(r_W)\mathrm{GL}(a)\mathrm{O}(p - p_1 - r_W - a, q - q_1 - r_W - a)$.

The group H_S must preserve $\mathbf{P}_U S$ and $(\mathbf{P}_U S)^\perp$ (restricted in U). So it preserves

$$\frac{\mathbf{P}_U S}{\mathrm{Rad}(\mathbf{P}_U S)} \quad \text{and} \quad \frac{(\mathbf{P}_U S)^\perp}{\mathrm{Rad}(\mathbf{P}_U S)}$$

which are of isometry types (defined by (7)) $(0, a_U, a_W)$ and $(0, p_1 - r_U - a - a_U, q_1 - r_U - a - a_W)$ respectively with respect to the induced bilinear form. Moreover, there is an orthogonal direct sum

$$\frac{\mathbf{P}_U S}{\mathrm{Rad}(\mathbf{P}_U S)} \oplus \frac{(\mathbf{P}_U S)^\perp}{\mathrm{Rad}(\mathbf{P}_U S)} = \frac{[\mathrm{Rad}(\mathbf{P}_U S)]^\perp}{\mathrm{Rad}(\mathbf{P}_U S)}. \tag{15}$$

Choose a basis $\mathcal{B}_U := \{\bar{\mathbf{u}}_1, \cdots, \bar{\mathbf{u}}_{p_1+q_1}\}$ of U such that each subspace in

$$U \cap S \subseteq \mathrm{Rad}(\mathbf{P}_U S) \subseteq \mathbf{P}_U S \subseteq [\mathrm{Rad}(\mathbf{P}_U S)]^\perp \subseteq (U \cap S)^\perp$$

is spanned by the first few vectors in $\mathcal{B}_U$. Then with respect to $\mathcal{B}_U$, the elements in $P_U(r_U, r_U + a)$ are of the form

$$h_1 = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} \\ 0 & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} \\ 0 & 0 & A_{33} & * & A_{35} & A_{36} \\ 0 & 0 & * & A_{44} & A_{45} & A_{46} \\ 0 & 0 & 0 & 0 & A_{55} & A_{56} \\ 0 & 0 & 0 & 0 & 0 & A_{66} \end{bmatrix}, \tag{16}$$

where $A_{11}, A_{66} \in \mathrm{GL}(r_U)$ uniquely determine each other, $A_{22}, A_{55} \in \mathrm{GL}(a)$ uniquely determine each other, and $\begin{bmatrix} A_{33} & * \\ * & A_{44} \end{bmatrix} \in \mathrm{O}(p_1 - r_U - a, q_1 - r_U - a)$. If $(h_1, h_2) \in H_S$ with $h_1 \in P_U(r_U, r_U + a)$ and $h_2 \in P_W(r_W, r_W + a)$, then h_1 must preserve (15). So $*$ in (16) must all vanish, and $A_{33} \times A_{44}$ is in the subgroup $\mathrm{O}(a_U, a_W) \times \mathrm{O}(p_1 - r_U - a - a_U, q_1 - r_U - a - a_W)$ of the $\mathrm{O}(p_1 - r_U - a, q_1 - r_U - a)$ factor.

Similarly, choose a basis $\mathcal{B}_W := \{\bar{\mathbf{w}}_1, \cdots, \bar{\mathbf{w}}_{p-p_1+q-q_1}\}$ of W such that each subspace of W in

$$W \cap S \subseteq \mathrm{Rad}(\mathbf{P}_W S) \subseteq \mathbf{P}_W S \subseteq [\mathrm{Rad}(\mathbf{P}_W S)]^\perp \subseteq (W \cap S)^\perp$$

is spanned by the first few vectors in $\mathcal{B}_W$, and furthermore $\bar{\mathbf{u}}_{r_U+i} + \bar{\mathbf{w}}_{r_W+i} \in S$ for $i = 1, 2, \ldots, a + a_U + a_W$. Then with respect to the basis $\mathcal{B}_W$, the elements in $P_W(r_W, r_W + a)$ are of the form

$$h_2 = \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} & B_{15} & B_{16} \\ 0 & B_{22} & B_{23} & B_{24} & B_{25} & B_{26} \\ 0 & 0 & B_{33} & * & B_{35} & B_{36} \\ 0 & 0 & * & B_{44} & B_{45} & B_{46} \\ 0 & 0 & 0 & 0 & B_{55} & B_{56} \\ 0 & 0 & 0 & 0 & 0 & B_{66} \end{bmatrix}, \tag{17}$$

where $B_{11}, B_{66} \in \mathrm{GL}(r_W)$ uniquely determine each other, $B_{22}, B_{55} \in \mathrm{GL}(a)$ uniquely determine each other, and $\begin{bmatrix} B_{33} & * \\ * & B_{44} \end{bmatrix} \in \mathrm{O}(p - p_1 - r_W - a, q - q_1 - r_W - a)$. Suppose that $(h_1, h_2) \in H_S$. Then $*$ in (17) must all vanish, and $B_{33} \times B_{44}$ is in the subgroup $\mathrm{O}(a_W, a_U) \times \mathrm{O}(p - p_1 - r_W - a - a_W, q - q_1 - r_W - a - a_U)$ of the $\mathrm{O}(p - p_1 - r_W - a, q - q_1 - r_W - a)$ factor. Moreover, $\begin{bmatrix} A_{22} & A_{23} \\ 0 & A_{33} \end{bmatrix} = \begin{bmatrix} B_{22} & B_{23} \\ 0 & B_{33} \end{bmatrix}$ since (h_1, h_2) must preserve the canonical bijection between $\dfrac{\mathbf{P}_U S}{U \cap S}$ and $\dfrac{\mathbf{P}_W S}{W \cap S}$ induced by S.

Theorem 3.1. *Suppose that* $(G, H) = (\mathrm{O}(p, q),\ \mathrm{O}(p_1, q_1) \times \mathrm{O}(p - p_1, q - q_1))$. *Let* $S \in \mathcal{O}_{G,H}(r_U, r_W, a, a_U, a_W)$. *Let* $h_1 \in P_U(r_U, r_U + a)$ *and* $h_2 \in P_W(r_W, r_W + a)$. *Then* $(h_1, h_2) \in H_S$ *if and only if*

(1) *the* $\mathrm{O}(p_1 - r_U - a,\ q_1 - r_U - a)$ *factor of* h_1 *in the Levi decomposition of* h_1 *is contained in a certain* $\mathrm{O}(a_U,\ a_W) \times \mathrm{O}(p_1 - r_U - a - a_U,\ q_1 - r_U - a - a_W)$;
(2) *the* $\mathrm{O}(p - p_1 - r_W - a,\ q - q_1 - r_W - a)$ *factor of* h_2 *in the Levi decomposition of* h_2 *is contained in a certain* $\mathrm{O}(a_W,\ a_U) \times \mathrm{O}(p - p_1 - r_W - a - a_W,\ q - q_1 - r_W - a - a_U)$;
(3) *the submatrices* $\begin{bmatrix} A_{22} & A_{23} \\ 0 & A_{33} \end{bmatrix}$ *of* h_1 *in* (16) *and* $\begin{bmatrix} B_{22} & B_{23} \\ 0 & B_{33} \end{bmatrix}$ *of* h_2 *in* (17) *are identical.*

In particular, $\dim H_S$ *can be computed by* (18) *below, and*

$$\dim \mathcal{O}_{G,H}(r_U, r_W, a, a_U, a_W) = \dim H - \dim H_S.$$

Proof. When $(h_1, h_2) \in H_S$, we have verified Theorem 3.1 (1)(2)(3). Conversely, if Theorem 3.1 (1)(2)(3) hold, then it is easy to see that $(h_1, h_2) \in H_S$.

It remains to compute $\dim H_S$. By the Levi decompositions, we have

$$\begin{aligned}\dim P_U(r_U, r_U+a) &= \frac{1}{2}[\dim \mathrm{O}(p_1,q_1)+\dim \mathrm{GL}(r_U)+\dim \mathrm{GL}(a) \\ &\qquad +\dim \mathrm{O}(p_1-r_U-a, q_1-r_U-a)], \\ \dim P_W(r_W, r_W+a) &= \frac{1}{2}[\dim \mathrm{O}(p-p_1,q-q_1)+\dim \mathrm{GL}(r_W)+\dim \mathrm{GL}(a) \\ &\qquad +\dim \mathrm{O}(p-p_1-r_W-a, q-q_1-r_W-a)].\end{aligned}$$

By Theorem 3.1 (1)(2)(3) and $r = r_U + r_W + a + a_U + a_W$, we have

$$\begin{aligned}\dim H_S &= \dim P_U(r_U, r_U+a)+\dim P_W(r_W, r_W+a) \\ &\quad -\dim \mathrm{O}(p_1-r_U-a, q_1-r_U-a)-\dim \mathrm{O}(p-p_1-r_W-a, q-q_1-r_W-a) \\ &\quad +\dim \mathrm{O}(a_U, a_W)+\dim \mathrm{O}(p_1-r_U-a-a_U, q_1-r_U-a-a_W) \\ &\quad +\dim \mathrm{O}(a_W, a_U)+\dim \mathrm{O}(p-p_1-r_W-a-a_W, q-q_1-r_W-a-a_U) \\ &\quad -\dim \mathrm{GL}(a)-\dim \mathrm{O}(a_U, a_W)-a(a_U+a_W) \\ &= \frac{1}{2}\binom{p_1+q_1}{2}+\frac{1}{2}\binom{p+q-p_1-q_1}{2}+\frac{r_U^2}{2}+\frac{r_W^2}{2}+\binom{a_U+a_W}{2}-a(a_U+a_W) \\ &\quad -\frac{1}{2}\binom{p_1+q_1-2r_U-2a}{2}-\frac{1}{2}\binom{p+q-p_1-q_1-2r_W-2a}{2} \\ &\quad +\binom{p_1+q_1-2r_U-2a-a_U-a_W}{2}+\binom{p+q-p_1-q_1-2r_W-2a-a_U-a_W}{2}. \qquad (18)\end{aligned}$$

□

The following corollary is a direct consequence of (18).

Corollary 3.2. *If both $\mathcal{O}_{G,H}(r_U, r_W, a, a_U, a_W)$ and $\mathcal{O}_{G,H}(r_U, r_W, a, a'_U, a'_W)$ exist and $a_U + a_W = a'_U + a'_W$, then they have the same dimension.*

Now we describe the open H-orbits in $\mathrm{Gr}_G(r)$. An H-orbit is open, if and only if it has the maximum dimension among all H-orbits in $\mathrm{Gr}_G(r)$, if and only if $\dim H_S$ is minimal for any subspace S in the G-orbit. The next theorem shows that to find the open H-orbit(s) $\mathcal{O}_{G,H}(r_U, r_W, a, a_U, a_W)$ in $\mathrm{Gr}_G(r)$, we should maximize a_U and a_W subject to the constraints $a_U + a_W \le r$, $a_U \le \min\{p_1, q-q_1\}$ and $a_W \le \min\{q_1, p-p_1\}$. In particular, $a = 0$ for all open orbits.

Theorem 3.3. *The open H-orbits in $\mathrm{Gr}_G(r)$ are determined as follow:*

(1) *When $r < \min\{p_1, q-q_1\} + \min\{q_1, p-p_1\}$, there are several open H-orbits and they are parametrized by*

$$\mathcal{O}_{G,H}(0,\ 0,\ 0,\ a_U,\ a_W)$$

with $a_U + a_W = r$, $0 \le a_U \le \min\{p_1, q-q_1\}$ and $0 \le a_W \le \min\{q_1, p-p_1\}$;

(2) *When* $\min\{p_1, q-q_1\} + \min\{q_1, p-p_1\} \leq r \leq \min\{p,q\}$, *there is a unique open* H*-orbit and it is parametrized by*

$$\begin{cases} \mathcal{O}_{G,H}(r-a_U-a_W,\ 0,\ 0,\ a_U,\ a_W) & \textit{if}\ \ \dim U > \dim W, \\ \mathcal{O}_{G,H}(0,\ r-a_U-a_W,\ 0,\ a_U,\ a_W) & \textit{if}\ \ \dim U \leq \dim W, \end{cases}$$

with $a_U = \min\{p_1, q-q_1\}$ *and* $a_W = \min\{q_1, p-p_1\}$.

Proof. Let $\mathcal{O}_{G,H}(r_U, r_W, a, a_U, a_W)$ be an open orbit and $S \in \mathcal{O}_{G,H}(r_U, r_W, a, a_U, a_W)$. Then the stabilizer H_S must be of minimal dimension among all $H_{S'}$ for $S' \in \mathrm{Gr}_G(r)$.

First, we prove that $a = 0$. If on the contrary $a > 0$, then $(r_U, r_W, 0, a_U + a, a_W)$ meets conditions (9) and there exists $S' \in \mathcal{O}_{G,H}(r_U, r_W, 0, a_U + a, a_W)$. By (18), we compute in MAPLE that

$$\dim H_S - \dim H_{S'} = \frac{1}{2}a(a+1) > 0.$$

Therefore $\dim H_S > \dim H_{S'}$. It contradicts the assumption that S is in an open orbit. Thus $a = 0$.

Second, we show that $r_U = 0$ or $r_W = 0$. If this is not true, then $r_U > 0$, $r_W > 0$, $a = 0$, and $r = r_U + r_W + a_U + a_W$. The 5-tuple $(r_U - 1, r_W - 1, 0, a_U + 1, a_W + 1)$ meets conditions (9) and thus there exists $S' \in \mathcal{O}_{G,H}(r_U - 1, r_W - 1, 0, a_U + 1, a_W + 1)$. By (18),

$$\dim H_S - \dim H_{S'} = p + q - r - a_U - a_W - 1 > p + q - 2r \geq 0.$$

It contradicts the assumption that H_S is of minimal dimension. Therefore $r_U = 0$ or $r_W = 0$.

Third, suppose that $r < \min\{p_1, q-q_1\} + \min\{q_1, p-p_1\}$. We claim $r_U = r_W = 0$. If not, without loss of generality, assume $r_U > 0$. Obviously, $r_W = 0$ and $a = 0$. By

$$a_U + a_W < r = r_U + a_U + a_W < \min\{p_1, q-q_1\} + \min\{q_1, p-p_1\}$$

we get $a_U < \min\{p_1, q-q_1\}$ or $a_W < \min\{q_1, p-p_1\}$. If $a_U < \min\{p_1, q-q_1\}$, then $(r_U - 1, 0, 0, a_U + 1, a_W)$ meets conditions (9) and there exists $S' \in \mathcal{O}_{G,H}(r_U - 1, 0, 0, a_U + 1, a_W)$. By (18),

$$\dim H_S - \dim H_{S'} = (p - p_1 + q - q_1 - a_U - a_W) + (r_U - 1) > 0.$$

If $a_W < \min\{q_1, p-p_1\}$, the same argument works for $S' \in \mathcal{O}_{G,H}(r_U - 1, 0, 0, a_U, a_W + 1)$. So H_S is not of the minimal dimension for $S \in \mathrm{Gr}_G(r)$. We reach a contradiction.

Therefore, when $r < \min\{p_1, q-q_1\} + \min\{q_1, p-p_1\}$, we have $r_U = r_W = a = 0$ and $a_U + a_W = r$. Moreover $a_U \leq \min\{p_1, q-q_1\}$ and $a_W \leq \min\{q_1, p-p_1\}$ by (9). This proves the necessary part of Theorem 3.3 (1). Conversely, all orbits with

$r_U = r_W = a = 0$ and $a_U + a_W = r$ have the same dimension by Corollary 3.2. So they are open orbits. This proves the sufficient part of Theorem 3.3 (1).

Finally, suppose $\min\{p, q\} \geq r \geq \min\{p_1, q - q_1\} + \min\{q_1, p - p_1\}$.

(A) Claim: $r_U = 0$ when $\dim U \leq \dim W$, and $r_W = 0$ when $\dim W \leq \dim U$.

Suppose that $\dim U \leq \dim W$. If on the contrary $r_U > 0$, then $r_W = 0$ and $a = 0$ by the preceding arguments. From (9b) and (9c),

$$2r_U + a_U + a_W \leq p_1 + q_1 = \dim U \leq \dim W = (q - q_1) + (p - p_1).$$

So $a_U < q-q_1$ or $a_W < p-p_1$. Without loss of generality, suppose $a_U < q-q_1$. Then $(r_U - 1, 0, 0, a_U + 1, a_W)$ meets conditions (9) and thus there exists $S' \in \mathcal{O}_{G,H}(r_U - 1, 0, 0, a_U + 1, a_W)$. By (18), we have

$$\dim H_S - \dim H_{S'} = (p - p_1 + q - q_1 - a_U - a_W) + (r_U - 1) > 0.$$

This contradicts the assumption that H_S is of minimal dimension. Therefore, $r_U = 0$ when $\dim U \leq \dim W$. Similarly, $r_W = 0$ when $\dim W \leq \dim U$.

(B) Claim: $a_U = \min\{p_1, q - q_1\}$ and $a_W = \min\{q_1, p - p_1\}$.

Without loss of generality, we assume that $\dim U \leq \dim W$. Then $r_U = 0$ and $a = 0$. Hence $r = r_W + a_U + a_W \geq \min\{p_1, q-q_1\} + \min\{q_1, p-p_1\}$. Conditions (9) show that $a_U \leq \min\{p_1, q-q_1\}$ and $a_W \leq \min\{q_1, p-p_1\}$. Suppose on the contrary $a_U < \min\{p_1, q - q_1\}$ (similar for $a_W < \min\{q_1, p - p_1\}$), then $r_W > 0$ and the 5-tuple $(0, r_W - 1, 0, a_U + 1, a_W)$ meets conditions (9). There exists $S' \in \mathcal{O}_{G,H}(0, r_W - 1, 0, a_U + 1, a_W)$. By (18),

$$\dim H_S - \dim H_{S'} = (p_1 + q_1 - a_U - a_W) + (r_W - 1) > 0.$$

This contradicts the assumption that H_S is of minimal dimension. Therefore, $a_U = \min\{p_1, q-q_1\}$ and $a_W = \min\{q_1, p-p_1\}$. Moreover, $r_W = r - a_U - a_W$.

The preceding argument proves the necessary part of Theorem 3.3 (2). Conversely, Theorem 3.3 (2) uniquely determines one H-orbit, which must be the open H-orbit of $\mathrm{Gr}_G(r)$. This proves the sufficient part. □

Example 3.4. *Suppose that*

$$0 < p_1, q_1 \leq r \leq p - p_1, q - q_1.$$

We describe the open H-orbits in $\mathrm{Gr}_G(r)$ by Theorem 3.3.

(1) *When $p_1, q_1 \leq r < p_1 + q_1 (= \dim U)$, the open orbits in $\mathrm{Gr}_G(r)$ are*

$$\mathcal{O}_{G,H}(0, 0, 0, a_U, r - a_U) \qquad \text{for} \qquad r - q_1 \leq a_U \leq p_1.$$

Totally, there are $(p_1 + q_1 + 1 - r)$ many open orbits in $\mathrm{Gr}_G(r)$.

(2) *When* $p_1+q_1 \le r \le p-p_1, q-q_1$, *the open orbit in* $\mathrm{Gr}_G(r)$ *is unique and it is*
$$\mathcal{O}_{G,H}(0, r-p_1-q_1, 0, p_1, q_1).$$

4. Real Orthogonal Case: Decompose an H-orbit into $(H\cap G_0)$-orbits

Throughout this section, let $p_1, q_1, p-p_1, q-q_1 > 0$ and $0 < r \le \min\{p,q\}$. The orthogonal group $G = \mathrm{O}(p,q)$ and its subgroup $H = \mathrm{O}(p_1,q_1)\times \mathrm{O}(p-p_1, q-q_1)$ have 4 and 16 connected components respectively. Let G_0 denote the identity component of G. We are interesting in the decomposition of an H-orbit in $\mathrm{Gr}_G(r)$ into $(H\cap G_0)$-orbits. This gives the parametrization of
$$(H\cap G_0)\backslash \mathrm{Gr}_G(r) \simeq (H\cap G_0)\backslash G_0/P$$
where P is a maximal parabolic subgroup of G_0.

It turns out that every H-orbit in $\mathrm{Gr}_G(r)$ may decompose into 1, 2, or 4 $(H\cap G_0)$-orbits. Moreover, an open H-orbit in $\mathrm{Gr}_G(r)$ is also an open $(H\cap G_0)$-orbit except for the case $p=q=r$, in which the unique open H-orbit in $\mathrm{Gr}_G(r)$ decomposes into 2 open $(H\cap G_0)$-orbits.

For the pair (G,H) acting on $V = U\oplus W$, we fix certain orthogonal bases $\{\mathbf{u}_1^+, \cdots, \mathbf{u}_{p_1}^+, \mathbf{u}_1^-, \cdots, \mathbf{u}_{q_1}^-\}$ of U and $\{\mathbf{w}_1^+, \cdots, \mathbf{w}_{p-p_1}^+, \mathbf{w}_1^-, \cdots, \mathbf{w}_{q-q_1}^-\}$ of W respectively, such that for all applicable indices,
$$(\mathbf{u}_i^+, \mathbf{u}_i^+) = 1, \qquad (\mathbf{u}_j^-, \mathbf{u}_j^-) = -1, \qquad (\mathbf{w}_t^+, \mathbf{w}_t^+) = 1, \qquad (\mathbf{w}_\ell^-, \mathbf{w}_\ell^-) = -1.$$
These vectors in the same order form a basis $\mathcal{B}_V$ of V. Denote the matrices
$$I_n^{(+)} := I_n \qquad \text{and} \qquad I_n^{(-)} := I_{n-1}\oplus(-I_1). \tag{19}$$
Then with respect to the basis $\mathcal{B}_V$, the 4 connected components of G are represented by elements $I_{p_1}^{(t_1)} \oplus I_{p_2}^{(t_2)} \oplus I_{p-p_1}^{(+)} \oplus I_{q-q_1}^{(+)}$ for $t_1, t_2 \in \{+,-\}$. and the 16 connected components of H can be denoted by $H_{t_3t_4}^{t_1t_2}$ where $t_1,t_2,t_3,t_4 \in \{+,-\}$ and
$$H_{t_3t_4}^{t_1t_2} \quad \ni \quad I_{p_1}^{(t_1)} \oplus I_{q_1}^{(t_2)} \oplus I_{p-p_1}^{(t_3)} \oplus I_{q-q_1}^{(t_4)}.$$
Moreover, $H\cap G_0$ consists of 4 components:
$$H\cap G_0 = H_{++}^{++} \cup H_{+-}^{+-} \cup H_{-+}^{-+} \cup H_{--}^{--}, \tag{20}$$
and $H/(H\cap G_0)$ consists of 4 cosets:
$$H/(H\cap G_0) = \left\{H_{++}^{++}(H\cap G_0), H_{++}^{+-}(H\cap G_0), H_{++}^{-+}(H\cap G_0), H_{++}^{--}(H\cap G_0)\right\}. \tag{21}$$

Let S spanned by (14) be the canonical subspace of an H-orbit $\mathcal{O}_{G,H}(r_U, r_W, a, a_U, a_W)$ in $\mathrm{Gr}_G(r)$. The $(H\cap G_0)$-orbit and the H-orbit of S are related by the canonical maps:
$$\begin{aligned}(H\cap G_0)\cdot S &\simeq (H\cap G_0)/(H\cap G_0)_S \\ &= (H\cap G_0)/[H_S\cap(H\cap G_0)] \\ &\simeq [H_S(H\cap G_0)]/H_S \\ &\subseteq H/H_S \;\simeq\; H\cdot S.\end{aligned} \tag{22}$$

So the number of $(H \cap G_0)$-orbits in the H-orbit of S is equal to $[H : H_S(H \cap G_0)]$, which by (21) is equal to $4/m$, and m is the number of cosets of $H/(H \cap G_0)$ — namely $H^{++}_{++}(H \cap G_0)$, $H^{+-}_{++}(H \cap G_0)$, $H^{-+}_{++}(H \cap G_0)$, and $H^{--}_{++}(H \cap G_0)$ — that intersect H_S. In particular, the H-orbit of S decomposes into only 1, 2, or 4 $(H \cap G_0)$-orbits.

The 5-tuple (r_U, r_W, a, a_U, a_W) satisfies conditions (9). The number of cosets of $H/(H \cap G_0)$ that intersect S can be determined by (r_U, r_W, a, a_U, a_W) as follow:

(1) Claim: If $r_U + a + a_U < p_1$ or $r_W + a + a_W < p - p_1$, then H_S intersects $H^{-+}_{++}(H \cap G_0)$.

If $r_W + a + a_W < p - p_1$ in (9d), then $\mathbf{w}^+_{p-p_1}$ does not appear in the basis (14) of S (i.e. $\mathbf{w}^+_{p-p_1} \in S^\perp$). Let $L \in \mathrm{GL}(V)$ have -1 eigenvector $\mathbf{w}^+_{p-p_1}$ and +1 eigenvectors in $\mathcal{B}_V \backslash \{\mathbf{w}^+_{p-p_1}\}$. Then $L \in H_S \cap H^{++}_{-+} \subseteq H_S \cap [H^{-+}_{++}(H \cap G_0)]$ since $H^{++}_{-+} = H^{-+}_{++} H^{-+}_{-+}$.

Similar for $r_U + a + a_U < p_1$ in (9b).

(2) Claim: If $r_U + a + a_W < q_1$ or $r_W + a + a_U < q - q_1$, then H_S intersects $H^{+-}_{++}(H \cap G_0)$.

The argument is similar to the preceding one.

(3) Claim: If $r_U > 0$ or $r_W > 0$ or $a_U > 0$ or $a_W > 0$, then H_S intersects $H^{--}_{++}(H \cap G_0)$.

If $a_U > 0$, then S has a basis vector $\mathbf{u}^+_i + \mathbf{w}^-_j$ in (14) for some i, j. Let $L \in \mathrm{GL}(V)$ have -1 eigenvectors in $\{\mathbf{u}^+_i, \mathbf{w}^-_j\}$ and +1 eigenvectors in $\mathcal{B}_V \backslash \{\mathbf{u}^+_i, \mathbf{w}^-_j\}$. Then $L \in H_S \cap H^{-+}_{+-} \subseteq H_S \cap [H^{--}_{++}(H \cap G_0)]$ since $H^{-+}_{+-} = H^{--}_{++} H^{+-}_{+-}$.

Similar for the other cases.

Obviously, H_S always intersects $H^{++}_{++}(H \cap G_0) = H \cap G_0$. These imply the following theorem.

Theorem 4.1. *The number N of $(H \cap G_0)$-orbits in an H-orbit is given below:*

N	*G*	*H*	*H-orbit*	*conditions*
4	$\mathrm{O}(2a, 2a)$	$\mathrm{O}(a, a) \times \mathrm{O}(a, a)$	$\mathcal{O}_{G,H}(0, 0, a, 0, 0)$	
2	$\mathrm{O}(p, 2a)$	$\mathrm{O}(p_1, a) \times \mathrm{O}(p - p_1, a)$	$\mathcal{O}_{G,H}(0, 0, a, 0, 0)$	$p_1 \geq a$, $p - p_1 \geq a$, $p > 2a$
	$\mathrm{O}(2a, q)$	$\mathrm{O}(a, q_1) \times \mathrm{O}(a, q - q_1)$	$\mathcal{O}_{G,H}(0, 0, a, 0, 0)$	$q_1 \geq a$, $q - q_1 \geq a$, $q > 2a$
	$\mathrm{O}(p, q)$ $(p = q)$	$\mathrm{O}(p_1, q_1) \times \mathrm{O}(p - p_1, q - q_1)$	$\mathcal{O}_{G,H}(r_U, r_W, a, a_U, a_W)$	$r_U + a + a_U = p_1$, $r_W + a + a_W = p - p_1$, $r_U + a + a_W = q_1$, $r_W + a + a_U = q - q_1$, $r_U + r_W + a_U + a_W > 0$
1	*all the other situations.*			

Theorem 4.1 together with Theorem 3.3 provides the following decomposition of an open H-orbit into open $(H \cap G_0)$-orbits.

Corollary 4.2. *An open H-orbit in $\mathrm{Gr}_G(r)$ always decomposes into 1 open $(H \cap G_0)$-orbit except for the case $p = q = r$, in which the unique open H-orbit decomposes into 2 open $(H \cap G_0)$-orbits.*

Proof. Let $\mathcal{O}_{G,H}(r_U, r_W, a, a_U, a_W)$ be an open H-orbit in $\mathrm{Gr}_G(r)$. By Theorem 3.3, we have $a = 0$ and $a_U + a_W > 0$. So by Theorem 4.1, $\mathcal{O}_{G,H}(r_U, r_W, a, a_U, a_W)$ decomposes into 1 or 2 open $(H\cap G_0)$-orbits, and it decomposes into 2 $(H\cap G_0)$-orbits only when

$$r_U + a_U = p_1, \quad r_W + a_W = p - p_1, \quad r_U + a_W = q_1, \quad r_W + a_U = q - q_1.$$

This together with $r = r_U + r_W + a_U + a_W$ implies that $p = q = r$. In such case, Theorem 3.3 shows that the open H-orbit is unique. □

Example 4.3. *We describe the open H-orbits that each decomposes into 2 open $(H\cap G_0)$-orbits for $\dim U \le \dim W$. In such case, $G = \mathrm{O}(r,r)$, $H = \mathrm{O}(p_1,q_1)\times \mathrm{O}(r-p_1, r-q_1)$, and $p_1+q_1 \le r$. The open H-orbit is $\mathcal{O}_{G,H}(0, r-p_1-q_1, 0, p_1, q_1)$ in $\mathrm{Gr}_G(r)$. The canonical subspace S of the H-orbit is spanned by (14):*

$$\left\{\mathbf{w}_j^+ + \mathbf{w}_j^-\right\}_{j=1}^{r-p_1-q_1} \;\cup\; \left\{\mathbf{u}_c^+ + \mathbf{w}_{r-p_1-q_1+c}^-\right\}_{c=1}^{p_1} \;\cup\; \left\{\mathbf{u}_d^- + \mathbf{w}_{r-p_1-q_1+d}^+\right\}_{d=1}^{q_1}.$$

We change the sign of one of the vectors $\mathbf{u}_s^\pm$ and $\mathbf{w}_t^\pm$ in the above basis vectors, and let S' be the subspace spanned by the new set of vectors. Then S and S' are in the same open H-orbit but in the two different open $(H\cap G_0)$-orbits.

5. Unitary Case

In this section, let

$$G := \mathrm{U}(p,q) \quad \text{and} \quad H := \mathrm{U}(p_1,q_1)\times \mathrm{U}(p-p_1, q-q_1),$$

where G is the isometry group of

$$V = U \oplus W \simeq \mathbb{C}^{p_1+q_1} \oplus \mathbb{C}^{p-p_1+q-q_1} \text{ (orthogonal direct sum)},$$

and H the subgroup of G that stabilizes U and W. We explore the H-action on the Grassmannian $\mathrm{Gr}_G(r)$ of r-dimensional isotropic subspaces of V for $r \le \min\{p,q\}$. Most results are completely parallel to those in real orthogonal cases (see Sections 2, 3, and 4). We sketch some proofs and omit other proofs that are routine. A distinction between unitary and real orthogonal cases is that the unitary groups here are connected and reductive, while their real orthogonal counterparts are disconnected and semisimple. However, the difference makes no impact in our analysis.

Theorem 5.1. *The H-orbit of an isotropic subspace $S\in \mathrm{Gr}_G(r)$ can be completely parametrized by the H-invariant 5-tuple (r_U, r_W, a, a_U, a_W) defined as in (8). The 5-tuple (r_U, r_W, a, a_U, a_W) satisfies the same constraints as in (9).*

Denote the H-orbit of S by $\mathcal{O}_{G,H}(r_U, r_W, a, a_U, a_W)$.

The stabilizer H_S has similar structure as in Section 3. An element of H_S can be written as (h_1, h_2) with $h_1 \in P_U(r_U, r_U + a)$ and $h_2 \in P_W(r_W, r_W + a)$, where $P_U(r_U, r_U+a)$ is the parabolic subgroup of $H|_U$ preserving the partial flag

$$U\cap S \subseteq \mathrm{Rad}(\mathbf{P}_U S) \subseteq [\mathrm{Rad}(\mathbf{P}_U S)]^\perp \subseteq (U\cap S)^\perp \quad \text{(all are restricted in } U\text{)},$$

and $P_W(r_W, r_W + a)$ is the parabolic subgroup of $H|_W$ preserving the partial flag

$$W \cap S \subseteq \mathrm{Rad}(\mathbf{P}_W S) \subseteq [\mathrm{Rad}(\mathbf{P}_W S)]^\perp \subseteq (W \cap S)^\perp \quad \text{(all are restricted in } W).$$

Moreover, h_1 preserves $\mathbf{P}_U S$, h_2 preserves $\mathbf{P}_W S$, and $(h_1, h_2) \in H_S$ preserves the canonical bijection (induced by S) between $\dfrac{\mathbf{P}_U S}{U \cap S}$ and $\dfrac{\mathbf{P}_W S}{W \cap S}$.

Let $\mathcal{B}_U := \{\bar{\mathbf{u}}_1, \cdots, \bar{\mathbf{u}}_{p_1+q_1}\}$ be a basis of U such that the subspaces in

$$U \cap S \subseteq \mathrm{Rad}(\mathbf{P}_U S) \subseteq \mathbf{P}_U S \subseteq [\mathrm{Rad}(\mathbf{P}_U S)]^\perp \subseteq (U \cap S)^\perp$$

are spanned by the first few vectors in $\mathcal{B}_U$. Let $\mathcal{B}_W := \{\bar{\mathbf{w}}_1, \cdots, \bar{\mathbf{w}}_{p-p_1+q-q_1}\}$ be a basis of W such that the subspaces in

$$W \cap S \subseteq \mathrm{Rad}(\mathbf{P}_W S) \subseteq \mathbf{P}_W S \subseteq [\mathrm{Rad}(\mathbf{P}_W S)]^\perp \subseteq (W \cap S)^\perp$$

are spanned by the first few vectors in $\mathcal{B}_W$, and furthermore $\bar{\mathbf{u}}_{r_U+i} + \bar{\mathbf{w}}_{r_W+i} \in S$ for $i = 1, 2, \cdots, a + a_U + a_W$. Then h_1 and h_2 have the following matrix representations with respect to $\mathcal{B}_U$ and $\mathcal{B}_W$ respectively:
(23)

$$h_1 = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} \\ 0 & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} \\ 0 & 0 & A_{33} & 0 & A_{35} & A_{36} \\ 0 & 0 & 0 & A_{44} & A_{45} & A_{46} \\ 0 & 0 & 0 & 0 & A_{55} & A_{56} \\ 0 & 0 & 0 & 0 & 0 & A_{66} \end{bmatrix}, \quad h_2 = \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} & B_{15} & B_{16} \\ 0 & B_{22} & B_{23} & B_{24} & B_{25} & B_{26} \\ 0 & 0 & B_{33} & 0 & B_{35} & B_{36} \\ 0 & 0 & 0 & B_{44} & B_{45} & B_{46} \\ 0 & 0 & 0 & 0 & B_{55} & B_{56} \\ 0 & 0 & 0 & 0 & 0 & B_{66} \end{bmatrix}.$$

Here $A_{11}, A_{66} \in \mathrm{GL}_{r_U}(\mathbb{C})$ uniquely determine each other; $A_{22}, A_{55} \in \mathrm{GL}_a(\mathbb{C})$ uniquely determine each other; $A_{33} \times A_{44}$ is in a subgroup $\mathrm{U}(a_U, a_W) \times \mathrm{U}(p_1 - r_U - a - a_U, q_1 - r_U - a - a_W)$ of the $\mathrm{U}(p_1 - r_U - a, q_1 - r_U - a)$ factor in the Levi decomposition of $P_U(r_U, r_U + a)$; $B_{11}, B_{66} \in \mathrm{GL}_{r_W}(\mathbb{C})$ uniquely determine each other; $B_{22}, B_{55} \in \mathrm{GL}_a(\mathbb{C})$ uniquely determine each other; $B_{33} \times B_{44}$ is in a subgroup $\mathrm{U}(a_W, a_U) \times \mathrm{U}(p - p_1 - r_W - a - a_W, q - q_1 - r_W - a - a_U)$ of the $\mathrm{U}(p - p_1 - r_W - a, q - q_1 - r_W - a)$ factor in the Levi decomposition of $P_W(r_W, r_W + a)$. Moreover $\begin{bmatrix} A_{22} & A_{23} \\ 0 & A_{33} \end{bmatrix} = \begin{bmatrix} B_{22} & B_{23} \\ 0 & B_{33} \end{bmatrix}$.

Theorem 5.2. *Let $S \in \mathcal{O}_{G,H}(r_U, r_W, a, a_U, a_W)$. Then $(h_1, h_2) \in H_S$ for $h_1 \in P_U(r_U, r_U + a)$ and $h_2 \in P_W(r_W, r_W + a)$ if and only if*

(1) *the $\mathrm{U}(p_1 - r_U - a, q_1 - r_U - a)$ factor in the Levi decomposition of h_1 is contained in a certain $\mathrm{U}(a_U, a_W) \times \mathrm{U}(p_1 - r_U - a - a_U, q_1 - r_U - a - a_W)$;*
(2) *the $\mathrm{U}(p - p_1 - r_W - a, q - q_1 - r_W - a)$ factor in the Levi decomposition of h_2 is contained in a certain $\mathrm{U}(a_W, a_U) \times \mathrm{U}(p - p_1 - r_W - a - a_W, q - q_1 - r_W - a - a_U)$;*
(3) *the submatrices $\begin{bmatrix} A_{22} & A_{23} \\ 0 & A_{33} \end{bmatrix}$ of h_1 and $\begin{bmatrix} B_{22} & B_{23} \\ 0 & B_{33} \end{bmatrix}$ of h_2 in (23) are equal.*

In particular, $\dim \mathcal{O}_{G,H}(r_U, r_W, a, a_U, a_W) = \dim H - \dim H_S$ *and*

$$\begin{aligned} \dim H_S &= (p_1+q_1)^2 + (p-p_1+q-q_1)^2 - 2(p_1+q_1)r_U - 2(p-p_1+q-q_1)r_W \\ &\quad + \frac{5}{2}r_U^2 + \frac{5}{2}r_W^2 + (a+a_U+a_W)(-2p-2q+4r-a_U-a_W). \end{aligned} \tag{24}$$

The above dimension of H_S is computed similarly as in real orthogonal case.

Corollary 5.3. *If both* $\mathcal{O}_{G,H}(r_U, r_W, a, a_U, a_W)$ *and* $\mathcal{O}_{G,H}(r_U, r_W, a, a'_U, a'_W)$ *exist and* $a_U + a_W = a'_U + a'_W$*, then they have the same dimension.*

Theorem 5.4. *The open H-orbits in* $\mathrm{Gr}_G(r)$ *are:*

(1) *When* $r < \min\{p_1, q-q_1\} + \min\{q_1, p-p_1\}$*, there are several open H-orbits and they are parametrized by*

$$\mathcal{O}_{G,H}(0,\ 0,\ 0,\ a_U,\ a_W)$$

with $a_U + a_W = r$, $0 \le a_U \le \min\{p_1, q-q_1\}$ *and* $0 \le a_W \le \min\{q_1, p-p_1\}$*;*

(2) *When* $\min\{p_1, q-q_1\} + \min\{q_1, p-p_1\} \le r \le \min\{p,q\}$*, there is a unique open H-orbit and it is parametrized by*

$$\begin{cases} \mathcal{O}_{G,H}(r-a_U-a_W,\ 0,\ 0,\ a_U,\ a_W) & \text{if } \dim U > \dim W, \\ \mathcal{O}_{G,H}(0,\ r-a_U-a_W,\ 0,\ a_U,\ a_W) & \text{if } \dim U \le \dim W, \end{cases}$$

with $a_U = \min\{p_1, q-q_1\}$ *and* $a_W = \min\{q_1, p-p_1\}$.

6. Orthogonal Case over $\mathbf{k}$

Let $G := \mathrm{O}_n(\mathbb{C})$ and $H := \mathrm{O}_m(\mathbb{C}) \times \mathrm{O}_{n-m}(\mathbb{C})$, where G is the isometry group of

$$V = U \oplus W \simeq \mathbb{C}^m \oplus \mathbb{C}^{n-m} \text{ (orthogonal direct sum)},$$

and H the subgroup stabilizing U and W. We study the H-action on $\mathrm{Gr}_G(r)$ ($r \le n/2$), the Grassmannian of r-dimensional isotropic subspaces of V. All results in this section may be extended to the orthogonal pairs $(G,H) = (\mathrm{O}_n(\mathbf{k}),\ \mathrm{O}_m(\mathbf{k}) \times \mathrm{O}_{n-m}(\mathbf{k}))$ over an algebraic closed field $\mathbf{k}$ of characteristic not 2.

Theorem 6.1. *The H-orbit of an isotropic subspace* $S \in \mathrm{Gr}_G(r)$ *can be completely parametrized by the H-invariant 4-tuple* (r_U, r_W, a, b) *defined by*

$$r_U := \dim(S \cap U), \tag{25a}$$

$$r_W := \dim(S \cap W), \tag{25b}$$

$$\begin{aligned} a &:= \dim(\mathrm{Rad}(\mathbf{P}_U S)) - r_U \\ &= \dim(\mathrm{Rad}(\mathbf{P}_W S)) - r_W, \end{aligned} \tag{25c}$$

$$\begin{aligned} b &:= \dim \mathbf{P}_U S - \dim(\mathrm{Rad}(\mathbf{P}_U S)) \\ &= \dim \mathbf{P}_W S - \dim(\mathrm{Rad}(\mathbf{P}_W S)). \end{aligned} \tag{25d}$$

The 4-tuple (r_U, r_W, a, b) *satisfies* $r_U + r_W + a + b = r$ *and the constraints:*

$$(r_U, r_W, a, b) \in \mathbb{N}_0{}^4, \tag{26a}$$

$$2r_U + 2a + b \leq m, \tag{26b}$$

$$2r_W + 2a + b \leq n - m. \tag{26c}$$

We denote the H*-orbit of* S *by* $\mathcal{O}_{G,H}(r_U, r_W, a, b)$.

Proof. The well-definedness of (r_U, r_W, a, b) by (25) is done similarly as in Theorem 2.1. Obviously (r_U, r_W, a, b) is H-invariant. We have $S \cap U \subseteq \mathrm{Rad}(\mathbf{P}_U S) \subseteq \mathbf{P}_U S \subseteq [\mathrm{Rad}(\mathbf{P}_U S)]^\perp$ in U. So

$$\begin{aligned} m - r_U - a &= \dim U - \dim \mathrm{Rad}(\mathbf{P}_U S) = \dim\{[\mathrm{Rad}(\mathbf{P}_U S)]^\perp \cap U\} \\ &\geq \dim \mathbf{P}_U S = r_U + a + b. \end{aligned}$$

So (26b) holds. Likewise (26c) holds.

Let S' be another isotropic subspace corresponding to the same 4-tuple (r_U, r_W, a, b). There exist orthonormal bases for $\frac{\mathbf{P}_U S}{\mathrm{Rad}(\mathbf{P}_U S)}$ and $\frac{\mathbf{P}_U S'}{\mathrm{Rad}(\mathbf{P}_U S')}$ since $\mathbb{C}$ is algebraic closed with characteristic not 2. So S and S' have decompositions

$$S = (S \cap U) \oplus \bigoplus_{i=1}^{t} \mathbb{C}(\mathbf{u}_i + \mathbf{w}_i), \qquad S' = (S' \cap U) \oplus \bigoplus_{i=1}^{t} \mathbb{C}(\mathbf{u}_i' + \mathbf{w}_i'),$$

where the Gram matrices

$$\begin{aligned} \big[\,(\mathbf{u}_i, \mathbf{u}_j)\,\big]_{t\times t} = \big[\,(\mathbf{u}_i', \mathbf{u}_j')\,\big]_{t\times t} &= 0_{a\times a} \oplus I_b, \\ \big[\,(\mathbf{w}_i, \mathbf{w}_j)\,\big]_{t\times t} = \big[\,(\mathbf{w}_i', \mathbf{w}_j')\,\big]_{t\times t} &= 0_{a\times a} \oplus (-I_b). \end{aligned}$$

Define an isometry $\phi : S \to S'$ that sends $S \cap U$ to $S' \cap U$ and $\mathbf{u}_i$ to $\mathbf{u}_i'$ for $i = 1, \cdots, t$. Then extend ϕ to an isometry g_1 of U by Witt's Theorem [17]. Similarly, there is an isometry g_2 of W that sends $S \cap W$ to $S' \cap W$ and $\mathbf{w}_i$ to $\mathbf{w}_i'$ for $i = 1, \cdots, t$. Then $g_1 \times g_2 \in H$ sends S to S'. This shows that the 4-tuple (r_U, r_W, a, b) uniquely determines one H-orbit.

Finally, given a 4-tuple (r_U, r_W, a, b) satisfying conditions (26), we shall prove that there exists an H-orbit corresponding to (r_U, r_W, a, b). Let $\mathcal{B}_U := \{\mathbf{u}_1, \cdots, \mathbf{u}_m\}$ be an orthonormal basis of U, and $\mathcal{B}_W := \{\mathbf{w}_1, \cdots, \mathbf{w}_{n-m}\}$ an orthonormal basis of W. Let S be the subspace spanned by the basis

$$\begin{aligned} &\{\mathbf{u}_i + \mathbf{i}\mathbf{u}_{m+1-i}\}_{i=1}^{r_U} \cup \{\mathbf{w}_j + \mathbf{i}\mathbf{w}_{n-m+1-j}\}_{j=1}^{r_W} \cup \{\mathbf{u}_{r_U+a+\ell} + \mathbf{i}\mathbf{w}_{r_W+a+\ell}\}_{\ell=1}^{b} \\ &\quad \cup \{\mathbf{u}_{r_U+k} + \mathbf{i}\mathbf{u}_{m+1-r_U-k} + \mathbf{w}_{r_W+k} + \mathbf{i}\mathbf{w}_{n-m+1-r_W-k}\}_{k=1}^{a} \end{aligned} \tag{27}$$

in which $\mathbf{i} := \sqrt{-1}$. Then S is a canonical isotropic subspace of V representing the H-orbit that corresponds to the 4-tuple (r_U, r_W, a, b). □

Now consider the stabilizer H_S. An element of H_S can be written as (h_1, h_2) with $h_1 \in P_U(r_U, r_U + a)$ and $h_2 \in P_W(r_W, r_W + a)$, where $P_U(r_U, r_U + a) \subseteq H|_U$ is the parabolic subgroup preserving the (partial) flag

$$U \cap S \subseteq \mathrm{Rad}(\mathbf{P}_U S) \subseteq [\mathrm{Rad}(\mathbf{P}_U S)]^{\perp} \subseteq (U \cap S)^{\perp} \quad \text{(all are restricted in } U\text{)},$$

and $P_W(r_W, r_W + a) \subseteq H|_W$ is the parabolic subgroup preserving the (partial) flag

$$W \cap S \subseteq \mathrm{Rad}(\mathbf{P}_W S) \subseteq [\mathrm{Rad}(\mathbf{P}_W S)]^{\perp} \subseteq (W \cap S)^{\perp} \quad \text{(all are restricted in } W\text{)}.$$

Moreover, h_1 preserves $\mathbf{P}_U S$, h_2 preserves $\mathbf{P}_W S$, and $h_1 \times h_2$ preserves the canonical bijection (induced by S) between $\dfrac{\mathbf{P}_U S}{U \cap S}$ and $\dfrac{\mathbf{P}_W S}{W \cap S}$.

Let $\mathcal{B}_U := \{\bar{\mathbf{u}}_1, \cdots, \bar{\mathbf{u}}_{p_1+q_1}\}$ be a basis of U such that each subspace in

$$U \cap S \subseteq \mathrm{Rad}(\mathbf{P}_U S) \subseteq \mathbf{P}_U S \subseteq [\mathrm{Rad}(\mathbf{P}_U S)]^{\perp} \subseteq (U \cap S)^{\perp}$$

is spanned by the first few vectors of $\mathcal{B}_U$. Let $\mathcal{B}_W := \{\bar{\mathbf{w}}_1, \cdots, \bar{\mathbf{w}}_{p-p_1+q-q_1}\}$ be a basis of W such that each subspace in

$$W \cap S \subseteq \mathrm{Rad}(\mathbf{P}_W S) \subseteq \mathbf{P}_W S \subseteq [\mathrm{Rad}(\mathbf{P}_W S)]^{\perp} \subseteq (W \cap S)^{\perp}$$

is spanned by the first few vectors of $\mathcal{B}_W$, and in addition $\bar{\mathbf{u}}_{r_U+i} + \bar{\mathbf{w}}_{r_W+i} \in S$ for $i = 1, 2, \cdots, a + b$. Then h_1 and h_2 have the following matrix representations with respect to $\mathcal{B}_U$ and $\mathcal{B}_W$ respectively:

(28)

$$h_1 = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} \\ 0 & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} \\ 0 & 0 & A_{33} & 0 & A_{35} & A_{36} \\ 0 & 0 & 0 & A_{44} & A_{45} & A_{46} \\ 0 & 0 & 0 & 0 & A_{55} & A_{56} \\ 0 & 0 & 0 & 0 & 0 & A_{66} \end{bmatrix}, \quad h_2 = \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} & B_{15} & B_{16} \\ 0 & B_{22} & B_{23} & B_{24} & B_{25} & B_{26} \\ 0 & 0 & B_{33} & 0 & B_{35} & B_{36} \\ 0 & 0 & 0 & B_{44} & B_{45} & B_{46} \\ 0 & 0 & 0 & 0 & B_{55} & B_{56} \\ 0 & 0 & 0 & 0 & 0 & B_{66} \end{bmatrix}.$$

Here $A_{11}, A_{66} \in \mathrm{GL}_{r_U}(\mathbb{C})$ uniquely determine each other; $A_{22}, A_{55} \in \mathrm{GL}_a(\mathbb{C})$ uniquely determine each other; $A_{33} \times A_{44}$ is in a subgroup $\mathrm{O}_b(\mathbb{C}) \times \mathrm{O}_{m-2r_U-2a-b}(\mathbb{C})$ of the $\mathrm{O}_{m-2r_U-2a}(\mathbb{C})$ factor in the Levi decomposition of $P_U(r_U, r_U+a)$; $B_{11}, B_{66} \in \mathrm{GL}_{r_W}(\mathbb{C})$ uniquely determine each other; $B_{22}, B_{55} \in \mathrm{GL}_a(\mathbb{C})$ uniquely determine each other; $B_{33} \times B_{44}$ is in a subgroup $\mathrm{O}_b(\mathbb{C}) \times \mathrm{O}_{n-m-2r_W-2a-b}(\mathbb{C})$ of the $\mathrm{O}_{n-m-2r_W-2a}(\mathbb{C})$ factor in the Levi decomposition of $P_W(r_W, r_W + a)$. Moreover $\begin{bmatrix} A_{22} & A_{23} \\ 0 & A_{33} \end{bmatrix} = \begin{bmatrix} B_{22} & B_{23} \\ 0 & B_{33} \end{bmatrix}$.

Theorem 6.2. *Let $S \in \mathcal{O}_{G,H}(r_U, r_W, a, b)$. Then $(h_1, h_2) \in H_S$ if and only if*

1. $h_1 \in P_U(r_U, r_U + a)$ *and* $h_2 \in P_W(r_W, r_W + a)$.
2. *the* $\mathrm{O}_{m-2r_U-2a}(\mathbb{C})$ *factor of* h_1 *in the Levi decomposition of* h_1 *is contained in a certain* $\mathrm{O}_b(\mathbb{C}) \times \mathrm{O}_{m-2r_U-2a-b}(\mathbb{C})$*;*
3. *the* $\mathrm{O}_{n-m-2r_W-2a}(\mathbb{C})$ *factor of* h_2 *in the Levi decomposition of* h_2 *is contained in a certain* $\mathrm{O}_b(\mathbb{C}) \times \mathrm{O}_{n-m-2r_W-2a-b}(\mathbb{C})$*;*

(4) *the submatrices* $\begin{bmatrix} A_{22} & A_{23} \\ 0 & A_{33} \end{bmatrix}$ *of* h_1 *and* $\begin{bmatrix} B_{22} & B_{23} \\ 0 & B_{33} \end{bmatrix}$ *of* h_2 *in* (28) *are equal.*

In particular, $\dim \mathcal{O}_{G,H}(r_U, r_W, a, b) = \dim H - \dim H_S$ *and*

$$\begin{aligned} \dim H_S \;&=\; \frac{1}{2}\binom{m}{2} + \frac{1}{2}\binom{n-m}{2} + \frac{r_U^2}{2} + \frac{r_W^2}{2} + \binom{b}{2} - ab \\ &\quad -\frac{1}{2}\binom{m-2r_U-2a}{2} - \frac{1}{2}\binom{n-m-2r_W-2a}{2} \\ &\quad +\binom{m-2r_U-2a-b}{2} + \binom{n-m-2r_W-2a-b}{2}. \end{aligned} \tag{29}$$

The expression of $\dim H_S$ is in the same form as in the real orthogonal case (18) except that we use b in place of $a_U + a_W$ here. We can similarly prove the following theorem regarding the open H-orbits (c.f. Theorem 3.3).

Theorem 6.3. *If* $r \le \min\{m, n-m\}$, *then the unique open* H*-orbit is* $\mathcal{O}_{G,H}(0,\ 0,\ 0,\ r)$; *if* $\min\{m, n-m\} \le r \le \frac{n}{2}$, *the unique open* H*-orbit is*

$$\begin{cases} \mathcal{O}_{G,H}(r-n+m,\ 0,\ 0,\ n-m) & \text{if } \dim U > \dim W, \\ \mathcal{O}_{G,H}(0,\ r-m,\ 0,\ m) & \text{if } \dim U \le \dim W. \end{cases}$$

Next we discuss how an H-orbit decomposes into $(H\cap G_0)$-orbits. Let $I_n^{(+)} := I_n$ and $I_n^{(-)} := I_{n-1} \oplus (-I_1)$ as in (19). The group $G = \mathrm{O}_n(\mathbb{C})$ has 2 connected components $G_0 = \mathrm{SO}_n(\mathbb{C})$ and $I_n^{(-)} G_0$. When $0 < m < n$, the subgroup $H = \mathrm{O}_m(\mathbb{C}) \times \mathrm{O}_{n-m}(\mathbb{C})$ has 4 connected components $H_{t_2}^{t_1}$ where $t_1, t_2 \in \{+, -\}$ and

$$H_{t_2}^{t_1} \quad \ni \quad I_m^{(t_1)} \oplus I_{n-m}^{(t_2)}.$$

The subgroup $H \cap G_0$ has 2 components H_+^+ and H_-^-; the quotient

$$H/(H\cap G_0) \;=\; \left\{H_+^+(H\cap G_0),\ H_+^-(H\cap G_0)\right\}.$$

Let S be the isotropic subspace of $\mathcal{O}_{G,H}(r_U, r_W, a, b)$ spanned by (27). By the same argument as in (22), the H-orbit of S decomposes into

$$[H : H_S(H\cap G_0)] \;\in\; \{1, 2\}$$

many $(H\cap G_0)$-orbits. Moreover, the H-orbit of S decomposes into 2 $(H\cap G_0)$-orbits if and only if H_S has no intersection with $H_+^-(H\cap G_0)$. This could be determined by the basis (27) of S.

Theorem 6.4. *Every* H*-orbit of an isotropic subspace decomposes into 1 or 2* $(H \cap G_0)$*-orbits. Moreover, the* H*-orbit* $\mathcal{O}_{G,H}(r_U, r_W, a, b)$ *decomposes into 2* $(H \cap G_0)$*-orbits if and only if the equalities* (26b) *and* (26c) *hold, that is,* $r + a = n/2$ *where* $r = r_U + r_W + a + b$.

The proof is similar to the discussion preceding Theorem 4.1 and is omitted.

Corollary 6.5. *An open H-orbit decomposes into 2 open $(H \cap G_0)$-orbits if and only if $r = n/2$.*

Finally, we show by one example in the orthogonal case that the canonical map $\widetilde{f}$ defined in (2) is neither surjective nor injective.

Example 6.6. *Let the symmetric pair $(G,\ H) = (\mathrm{O}_8(\mathbb{C}),\ \mathrm{O}_4(\mathbb{C}) \times \mathrm{O}_4(\mathbb{C}))$ act on $V = U \oplus W = \mathbb{C}^4 \oplus \mathbb{C}^4$. Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ and $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4\}$ be orthonormal bases of U and W respectively. Let $\mathcal{F}'_G$ denote the partial flag variety of isotropic subspaces of dimensions 1 and 2. Define the canonical map*

$$\widetilde{f} : H\backslash\mathcal{F}'_G \to H\backslash\mathrm{Gr}_G(1) \times H\backslash\mathrm{Gr}_G(2).$$

(1) *($\widetilde{f}$ is not surjective) Let*

$$T_1 = \mathbb{C}(\mathbf{u}_1 + \mathbf{i}\mathbf{u}_2) \qquad \text{and} \qquad T_2 = \mathbb{C}(\mathbf{w}_1 + \mathbf{i}\mathbf{w}_2) + \mathbb{C}(\mathbf{w}_3 + \mathbf{i}\mathbf{w}_4).$$

Then $H \cdot T_1$ consists of isotropic lines in U and $H \cdot T_2$ consists of isotropic 2-dimensional subspaces in W. So $(H \cdot T_1, H \cdot T_2)$ is not in the image of $\widetilde{f}$ since there is no flag $\{C_1 \subset C_2\}$ in $\mathcal{F}'_G$ such that $C_1 \in H \cdot T_1$ and $C_2 \in H \cdot T_2$. Therefore $\widetilde{f}$ is not surjective.

(2) *($\widetilde{f}$ is not injective) Denote the isotropic subspaces*

$$\begin{aligned} S_1 = S'_1 &= \mathbb{C}(\mathbf{u}_1 + \mathbf{i}\mathbf{u}_2 + \mathbf{i}\mathbf{w}_1 - \mathbf{w}_2), \\ S_2 &= S_1 + \mathbb{C}(\mathbf{u}_1 - \mathbf{i}\mathbf{u}_2 + \mathbf{i}\mathbf{w}_1 + \mathbf{w}_2) + \mathbb{C}(\mathbf{u}_3 - \mathbf{i}\mathbf{u}_4 + \mathbf{i}\mathbf{w}_3 + \mathbf{w}_4), \\ S'_2 &= S'_1 + \mathbb{C}(\mathbf{u}_3 + \mathbf{i}\mathbf{u}_4 + \mathbf{i}\mathbf{w}_3 - \mathbf{w}_4) + \mathbb{C}(\mathbf{u}_3 - \mathbf{i}\mathbf{u}_4 + \mathbf{i}\mathbf{w}_3 + \mathbf{w}_4). \end{aligned}$$

Then S_1 and S'_1 are in the same H-orbit, and S_2 and S'_2 are in the same H-orbit by Theorem 6.1. Consider the following two flags in $\mathcal{F}'_G$:

$$F = \{S_1 \subset S_2\}, \qquad F' = \{S'_1 \subset S'_2\}.$$

We see that $\widetilde{f}(F) = \widetilde{f}(F')$.

If F and F' are in the same H-orbit, then there is $h \in H$ such that $h \cdot S_i = S'_i$ for $i = 1, 2$. Then

$$h \cdot [\mathbf{P}_U S_1 \cap (\mathbf{P}_U S_2)^\perp] = \mathbf{P}_U S'_1 \cap (\mathbf{P}_U S'_2)^\perp.$$

However, $\mathbf{P}_U S_1 \cap (\mathbf{P}_U S_2)^\perp = \{\mathbf{0}\}$ but $\mathbf{P}_U S'_1 \cap (\mathbf{P}_U S'_2)^\perp = \mathbf{P}_U S'_1 = \mathbb{C}(\mathbf{u}_1 + \mathbf{i}\mathbf{u}_2)$. We reach a contradiction. Therefore, F and F' are not in the same H-orbit. This together with $\widetilde{f}(F) = \widetilde{f}(F')$ shows that $\widetilde{f}$ is not injective.

7. Symplectic Case

Let V be a $2n$-dimensional real vector space equipped with a symplectic form $(\, , \,)$. Let G or sometimes $G(2n)$ be the symplectic group preserving $(\, , \,)$. Let H be the stabilizer of two subspaces U and W of V, where

$$V = U \oplus W, \qquad U \perp W, \qquad H|_U \cong G(2m), \qquad H|_W \cong G(2n-2m). \tag{30}$$

Let S be a r-dimensional isotropic subspace of V. Obviously, $r \leq n$. Denote the r-dimensional isotropic Grassmannian by $\mathrm{Gr}_G(r)$. Consider the H action in $\mathrm{Gr}_G(r)$. By Matsuki's theorem, there are only finite number of H-orbits in $\mathrm{Gr}_G(r)$. We would like to give an elementary parametrization of these orbits, compute their dimensions and describe the stabilizers. When $r = n$, H action on the Lagrangian Grassmannian $\mathrm{Gr}_G(n)$ is described in [2].

The structure of S in reference to the decomposition $U \oplus W$ can be described as follows:

$$S = (S \cap U) \oplus (S \cap W) \oplus \bigoplus_{i=1}^{k} \mathbb{R}(\mathbf{u}_i + \mathbf{w}_i),$$

with $\mathbf{u}_i \in U$ and $\mathbf{w}_i \in W$. In particular,

$$\mathbf{P}_U S \;:=\; (S + W) \cap U = (S \cap U) \oplus \bigoplus_{i=1}^{k} \mathbb{R}\mathbf{u}_i, \tag{31}$$

$$\mathbf{P}_W S \;:=\; (S + U) \cap W = (S \cap W) \oplus \bigoplus_{i=1}^{k} \mathbb{R}\mathbf{w}_i. \tag{32}$$

Notice that these decompositions are NOT cannonical. The vectors $\mathbf{u}_i$ and $\mathbf{w}_i$ are by no means unique. Observe that $\bigoplus_{i=1}^{k} \mathbb{R}\mathbf{u}_i$ can be decomposed into two subspaces: a radical and a nondegenerate subspace, with respect to $(\, , \,)$.

Theorem 7.1. *Let S be an isotropic subspace of $V = U \oplus W$. Let*

$$\begin{aligned} r_U &= \dim(S \cap U), \\ r_W &= \dim(S \cap W), \\ a &= \dim(\mathrm{Rad}(\mathbf{P}_U S)) - r_U, \\ b &= \dim(\mathbf{P}_U S) - a - r_U. \end{aligned}$$

Then b must be even, $a + b + r_U + r_W = \dim S = r$ and

$$a = \dim(\mathrm{Rad}(\mathbf{P}_W S)) - r_W, \qquad b = \dim(\mathbf{P}_W S) - a - r_W.$$

The 4-tuple (r_U, r_W, a, b) uniquely determines the H-orbit of S. In addition, an isotropic S parametrized by the 4-tuple $(r_U, r_W, a, b) \in {\mathbb{N}_0}^4$ exists if and only if b

is even and

$$r_U + a + \frac{b}{2} \leq m, \tag{33a}$$

$$r_W + a + \frac{b}{2} \leq n - m. \tag{33b}$$

Proof. Consider the restriction of the symplectic form on $\bigoplus_{i=1}^k \mathbb{R}\mathbf{u}_i$. In reference to (31), we have

$$\mathrm{Rad}(\mathbf{P}_U S) = \mathrm{Rad}(\bigoplus_{i=1}^{k} \mathbb{R}\mathbf{u}_i) \oplus (S \cap U).$$

So $a := \dim(\mathrm{Rad}(\mathbf{P}_U S)) - r_U = \dim(\mathrm{Rad}(\bigoplus \mathbb{R}\mathbf{u}_i))$. Hence $b := \dim(\mathbf{P}_U S) - a - r_U = \dim[(\bigoplus \mathbb{R}\mathbf{u}_i)/\mathrm{Rad}(\bigoplus \mathbb{R}\mathbf{u}_i)]$ must be even dimensional. In addition

$$r = \dim(S) = \dim(S \cap U) + \dim(S \cap W) + k = r_U + r_W + a + b.$$

Define a linear isomorphism Φ from $\bigoplus_{i=1}^k \mathbb{R}\mathbf{u}_i$ to $\bigoplus_{i=1}^k \mathbb{R}\mathbf{w}_i$ by letting $\Phi(\mathbf{u}_i) = \mathbf{w}_i$ for every i. Since $\bigoplus_{i=1}^k \mathbb{R}(\mathbf{u}_i + \mathbf{w}_i)$ is isotropic, Φ must negate the symplectic form $(,)$ restricted to the two linear subspaces. Hence

$$\Phi(\mathrm{Rad}(\bigoplus_{i=1}^{k} \mathbb{R}\mathbf{u}_i)) = \mathrm{Rad}(\bigoplus_{i=1}^{k} \mathbb{R}\mathbf{w}_i).$$

So $a = \dim(\mathrm{Rad}(\mathbf{P}_W S)) - r_W$ and $b = \dim(\mathbf{P}_W S) - a - r_W$. In addition, a Lagrangian subspace of $(\bigoplus \mathbb{R}\mathbf{u}_i)/\mathrm{Rad}(\bigoplus \mathbb{R}\mathbf{u}_i)$ induces an isotropic subspace in U by adding $\mathrm{Rad}(\mathbf{P}_U S)$. We have $r_U + a + \frac{b}{2} \leq m$. Similarly, we have $r_W + a + \frac{b}{2} \leq n - m$.

Now given (r_U, r_W, a, b) satisfying the conditions (33), we construct an r-dimensional isotropic subspace as follows. Fix standard bases $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m, \mathbf{f}_1, \mathbf{f}_2, \ldots \mathbf{f}_m\}$ in U and $\{\mathbf{e}_{m+1}, \mathbf{e}_{m+2}, \ldots, \mathbf{e}_n, \mathbf{f}_{m+1}, \mathbf{f}_{m+2}, \ldots \mathbf{f}_n\}$ in W. Let S be the canonical isotropic subspace spanned by

$$\{\mathbf{e}_i\}_{i=m-r_U+1}^{m} \cup \{\mathbf{e}_j\}_{j=n-r_W+1}^{n} \cup \{\mathbf{e}_\ell + \mathbf{f}_{m+\ell},\ \mathbf{f}_\ell + \mathbf{e}_{m+\ell}\}_{\ell=1}^{b/2} \cup \{\mathbf{e}_k + \mathbf{e}_{m+k}\}_{k=b/2+1}^{b/2+a}. \tag{34}$$

It is clear that S represents the H-orbit parametrized by (r_U, r_W, a, b).

It remains to show that (r_U, r_W, a, b) parametrizes a single H-orbit. This can be done similarly as in the proof of Theorem 2.1. □

Suppose that b is an even nonnegative integer. Suppose that $r_U + a + \frac{b}{2} \leq m$ and $r_W + a + \frac{b}{2} \leq n - m$. Let $\mathcal{O}_{G,H}(r_U, r_W, a, b)$ or $\mathcal{O}(r_U, r_W, a, b)$ be the H-orbit parametrized by (r_U, r_W, a, b). Let $r = r_U + r_W + a + b$. Then $\mathcal{O}(r_U, r_W, a, b) \subseteq \mathrm{Gr}_G(r)$.

Now fix $S \in \mathcal{O}(r_U, r_W, a, b)$. We would like to compute its stabilizer H_S. Let $P_U(r_U, r_U + a)$ be the parabolic subgroup of $G(U)$ preserving the (partial) flag

$$U \cap S \subseteq \mathrm{Rad}(\mathbf{P}_U S) \subseteq \mathrm{Rad}(\mathbf{P}_U S)^\perp \subseteq (U \cap S)^\perp \quad \text{(all are restricted in } U),$$

and $P_W(r_W, r_W + a)$ the parabolic subgroup of $G(W)$ preserving the (partial) flag

$$W \cap S \subseteq \mathrm{Rad}(\mathbf{P}_W S) \subseteq \mathrm{Rad}(\mathbf{P}_W S)^\perp \subseteq (W \cap S)^\perp \quad \text{(all are restricted in } W).$$

The Levi factor of $P_U(r_U, r_U + a)$ is $\mathrm{GL}(r_U)\mathrm{GL}(a)G(2m - 2r_U - 2a)$ and the Levi factor of $P_W(r_W, r_W + a)$ is $\mathrm{GL}(r_W)\mathrm{GL}(a)G(2n - 2m - 2r_W - 2a)$.

With the settings, an element of H_S can be written as (h_1, h_2) for $h_1 \in P_U(r_U, r_U + a)$ and $h_2 \in P_W(r_W, r_W + a)$. Moreover, h_1 preserves $\mathbf{P}_U S$, h_2 preserves $\mathbf{P}_W S$, and $h_1 \times h_2$ preserves the canonical bijection (induced by S) between $\dfrac{\mathbf{P}_U S}{U \cap S}$ and $\dfrac{\mathbf{P}_W S}{W \cap S}$. Let $\mathcal{B}_U := \{\bar{\mathbf{u}}_1, \cdots, \bar{\mathbf{u}}_{p_1+q_1}\}$ be a basis of U such that each subspace in

$$U \cap S \subseteq \mathrm{Rad}(\mathbf{P}_U S) \subseteq \mathbf{P}_U S \subseteq [\mathrm{Rad}(\mathbf{P}_U S)]^\perp \subseteq (U \cap S)^\perp$$

is spanned by the first few vectors of $\mathcal{B}_U$. Let $\mathcal{B}_W := \{\bar{\mathbf{w}}_1, \cdots, \bar{\mathbf{w}}_{p-p_1+q-q_1}\}$ be a basis of W such that each subspace in

$$W \cap S \subseteq \mathrm{Rad}(\mathbf{P}_W S) \subseteq \mathbf{P}_W S \subseteq [\mathrm{Rad}(\mathbf{P}_W S)]^\perp \subseteq (W \cap S)^\perp$$

is spanned by the first few vectors of $\mathcal{B}_W$, and in addition $\bar{\mathbf{u}}_{r_U+i} + \bar{\mathbf{w}}_{r_W+i} \in S$ for $i = 1, 2, \cdots, a + b$. Then h_1 and h_2 have the following matrix representations with respect to $\mathcal{B}_U$ and $\mathcal{B}_W$ respectively:

(35)

$$h_1 = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} \\ 0 & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} \\ 0 & 0 & A_{33} & 0 & A_{35} & A_{36} \\ 0 & 0 & 0 & A_{44} & A_{45} & A_{46} \\ 0 & 0 & 0 & 0 & A_{55} & A_{56} \\ 0 & 0 & 0 & 0 & 0 & A_{66} \end{bmatrix}, \quad h_2 = \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} & B_{15} & B_{16} \\ 0 & B_{22} & B_{23} & B_{24} & B_{25} & B_{26} \\ 0 & 0 & B_{33} & 0 & B_{35} & B_{36} \\ 0 & 0 & 0 & B_{44} & B_{45} & B_{46} \\ 0 & 0 & 0 & 0 & B_{55} & B_{56} \\ 0 & 0 & 0 & 0 & 0 & B_{66} \end{bmatrix}.$$

Here $A_{11}, A_{66} \in \mathrm{GL}_{r_U}(\mathbb{C})$ uniquely determine each other; $A_{22}, A_{55} \in \mathrm{GL}_a(\mathbb{C})$ uniquely determine each other; $A_{33} \times A_{44}$ is in a subgroup $G(b) \times G(2m - 2r_U - 2a - b)$ of the $G(2m - 2r_U - 2a)$ factor in the Levi decomposition of $P_U(r_U, r_U + a)$; $B_{11}, B_{66} \in \mathrm{GL}_{r_W}(\mathbb{C})$ uniquely determine each other; $B_{22}, B_{55} \in \mathrm{GL}_a(\mathbb{C})$ uniquely determine each other; $B_{33} \times B_{44}$ is in a subgroup $G(b) \times G(2n - 2m - 2r_W - 2a - b)$ of the $G(2n - 2m - 2r_W - 2a)$ factor in the Levi decomposition of $P_W(r_W, r_W + a)$. Moreover $\begin{bmatrix} A_{22} & A_{23} \\ 0 & A_{33} \end{bmatrix} = \begin{bmatrix} B_{22} & B_{23} \\ 0 & B_{33} \end{bmatrix}$.

Theorem 7.2. *Let $S \in \mathcal{O}(r_U, r_W, a, b)$. Let $h_1 \in P_U(r_U, r_U + a)$ and $h_2 \in P_W(r_W, r_W + a)$. Then $(h_1, h_2) \in H_S$ if and only if*

(1) *the* $G(2m-2r_U-2a)$ *factor in the Levi decomposition of* h_1 *is contained in a certain* $G(b)\times G(2m-2r_U-2a-b)$*;*
(2) *the* $G(2n-2m-2r_W-2a)$ *factor in the Levi decomposition of* h_2 *is contained in a certain* $G(b)\times G(2n-2m-2r_W-2a-b)$*;*
(3) *the submatrices* $\begin{bmatrix} A_{22} & A_{23} \\ 0 & A_{33}\end{bmatrix}$ *of* h_1 *and* $\begin{bmatrix} B_{22} & B_{23} \\ 0 & B_{33}\end{bmatrix}$ *of* h_2 *in* (35) *are identical.*

In particular, $\dim \mathcal{O}(r_U, r_W, a, b) = \dim H - \dim H_S$*, and*

$$\begin{aligned}\dim H_S &= \frac{1}{2}\dim G(2m) + \frac{1}{2}\dim G(2n-2m) + \frac{r_U^2+r_W^2}{2} - ab \\ &- \frac{1}{2}\dim G(2m-2r_U-2a) - \frac{1}{2}\dim G(2n-2m-2r_W-2a) + \dim G(b) \\ &+ \dim G(2m-2r_U-2a-b) + \dim G(2n-2m-2r_W-2a-b). \end{aligned} \tag{36}$$

Proof. The preceding discussion has shown that every $(h_1, h_2) \in H_S$ must satisfy Theorem 7.2 (1)(2)(3). Conversely, if $(h_1, h_2) \in P_U(r_U, r_U + a) \times P_W(r_W, r_W + a)$ satisfies Theorem 7.2 (1)(2)(3), then (h_1, h_2) preserves S by a direct computation.

It remains to compute $\dim H_S$. Notice that

$$\begin{aligned}\dim P_U(r_U, r_U+a) &= \frac{1}{2}\left[\dim G(2m) + r_U^2 + a^2 + \dim G(2m-2r_U-2a)\right], \\ \dim P_W(r_W, r_W+a) &= \frac{1}{2}\left[\dim G(2n-2m) + r_W^2 + a^2 + \dim G(2n-2m-2r_W-2a)\right].\end{aligned}$$

Taking into the consideration of Theorem 7.2 (1)(2)(3), we have

$$\begin{aligned}\dim H_S =\ & \dim P_U(r_U, r_U+a) + \dim P_W(r_W, r_W+a) \\ & - \dim G(2m-2r_U-2a) - \dim G(2n-2m-2r_W-2a) \\ & + \dim G(2m-2r_U-2a-b) + \dim G(2n-2m-2r_W-2a-b) \\ & + \dim G(b) - a^2 - ab.\end{aligned} \tag{37}$$

(36) then follows. ∎

Lemma 7.3. *Suppose that* $r_U + r_W + a + b = r$ *and inequalities* (33) *hold. If* $\mathcal{O}(r_U, r_W, a, b)$ *is open in* $\mathrm{Gr}_G(r)$ *then*

$$ar_U = 0, \qquad ar_W = 0, \qquad r_U r_W = 0, \qquad a \leq 1$$

Proof. $\mathcal{O}(r_U, r_W, a, b)$ is open, if and only if $\mathcal{O}(r_U, r_W, a, b)$ is of maximal dimension, if and only if H_S is of minimal dimension.

Suppose that $ar_U \neq 0$. Then $a \geq 1$ and $r_U \geq 1$. Notice

$$a-1+r_U-1+r_W+b+2 = r, \quad r_U-1+a-1+\frac{b+2}{2} \leq m, \quad r_W+a-1+\frac{b+2}{2} \leq n-m.$$

Let $S' \in \mathcal{O}(r_U - 1, r_W, a-1, b+2)$. By (36), we compute by MAPLE that

$$\dim(H_S) - \dim(H_{S'}) = r_U + 2(n - m - r_W - 1 - \frac{b}{2}) > 0.$$

We see that H_S is not of minimal dimension.

Similarly if $ar_W \neq 0$, then H_S is not of minimal dimension. Now suppose that $r_U r_W \neq 0$. Then $r_U \geq 1$ and $r_W \geq 1$. Notice that

$$r_U - 1 + r_W - 1 + a + b + 2 = r, \quad r_U - 1 + a + \frac{b+2}{2} \leq m, \quad r_W - 1 + a + \frac{b+2}{2} \leq n - m.$$

Let $S' \in \mathcal{O}(r_U - 1, r_W - 1, a, b+2)$. By (36), we have

$$\dim(H_S) - \dim(H_{S'}) = (r_U + r_W - 1) + 2(n - r_U - r_W - a - b) > 0.$$

So H_S is not of minimal dimension.

Suppose that $a \geq 2$. Then

$$r_U + a - 2 + \frac{b+2}{2} \leq m, \quad r_W + a - 2 + \frac{b+2}{2} \leq n - m, \quad r_U + r_W + a - 2 + b + 2 = r.$$

Let $S' \in \mathcal{O}(r_U, r_W, a-2, b+2)$. By (36), we have

$$\dim(H_S) - \dim(H_{S'}) = 2a - 3 > 0.$$

So H_S is not of minimal dimension. □

Lemma 7.4. *If $\mathcal{O}(r_U, r_W, a, b)$ is an open orbit, then b must be even and maximal.*

Proof. We have shown in Theorem 7.1 that b is even. Lemma 7.3 indicates that at most one of r_U, r_W, a is nonzero and $a \leq 1$. If on the contrary, b is not maximal, then $r_U \geq 2$ or $r_W \geq 2$. Let us say $r_U \geq 2$. Then $r_W = a = 0$ and $S \in \mathcal{O}(r_U, 0, 0, b)$. There are 3 possibilities for b not maximal:

(1) There exists an H-orbit $\mathcal{O}(r_U - 2x, 0, 0, b + 2x)$ for some $x \in \mathbf{Z}^+$. Let $S' \in \mathcal{O}(r_U - 2x, 0, 0, b + 2x)$. Then $n - m > \frac{1}{2}(b + 2x) = \frac{b}{2}$ and

$$\dim H_S - \dim H_{S'} = 4x(n - m - \frac{b}{2}) + 2x(r_U - 2x) > 0.$$

It contradicts the assumption that S is in an open H-orbit.

(2) There exists an H-orbit $\mathcal{O}(0, r_U - 2x, 0, b + 2x)$ for some $x \in \mathbf{Z}^+$. Then (33) implies that $\mathcal{O}(r_U - 2x, 0, 0, b + 2x)$ is also an H-orbit. Again we reach a contradiction.

(3) There exists an H-orbit $\mathcal{O}(0, 0, 1, r_U + b - 1)$ (where $r_U > 1$ is odd). Let $S' \in \mathcal{O}(0, 0, 1, r_U + b - 1)$. Then $n - m \geq \frac{1}{2}(r_U + b - 1) > \frac{b}{2}$ and

$$\dim H_S - \dim H_{S'} = 2r_U(n - m - \frac{b}{2}) > 0.$$

This contradicts that S is in an open H-orbit.

Therefore, b must be maximal in an open H-orbit. □

Theorem 7.5. *Let $(G, H) = (G(2n), G(2m) \times G(2n-2m))$.*

(1) *When $r \geq \min(2m, 2n-2m)$, the unique open H-orbit in $\mathrm{Gr}_G(r)$ is*

$$\begin{cases} \mathcal{O}(0, r-2m, 0, 2m) & \text{if } m \leq n-m; \\ \mathcal{O}(r-2n+2m, 0, 0, 2n-2m) & \text{if } m \geq n-m. \end{cases}$$

(2) *When $r \leq \min(2m, 2n-2m)$, the unique open H-orbit in $\mathrm{Gr}_G(r)$ is*

$$\begin{cases} \mathcal{O}(0, 0, 0, r) & \text{if } r \text{ is even;} \\ \mathcal{O}(0, 0, 1, r-1) & \text{if } r \text{ is odd.} \end{cases}$$

Proof. By Lemma 7.3 and Lemma 7.4, it remains to prove that if $\mathcal{O}(0,0,1,r-1)$ is an H-orbit — in which $r \leq \min(2m, 2n-2m)$ by (33) and r is odd — then its dimension is greater than those of $\mathcal{O}(1,0,0,r-1)$ and $\mathcal{O}(0,1,0,r-1)$. Let $S \in \mathcal{O}(1,0,0,r-1)$ and $S' \in \mathcal{O}(0,0,1,r-1)$. Then by (33),

$$\dim H_S - \dim H_{S'} = (2n-2m-r) + 1 > 0.$$

So $\dim \mathcal{O}(0,0,1,r-1) > \dim \mathcal{O}(1,0,0,r-1)$. Likewise, $\dim \mathcal{O}(0,0,1,r-1) > \dim \mathcal{O}(0,1,0,r-1)$. □

Example 7.6. *Let $G = G(4m+8)$ and $H = G(2m) \times G(2m+8)$. The open H-orbits in $\mathrm{Gr}_G(r)$ for increasing r are:*

$$\mathcal{O}(0,0,1,0),\ \mathcal{O}(0,0,0,2),\ \mathcal{O}(0,0,1,2),\ \mathcal{O}(0,0,0,4),\ \cdots,\ \mathcal{O}(0,0,1,2m-2),\ \mathcal{O}(0,0,0,2m),$$
$$\mathcal{O}(0,1,0,2m),\ \mathcal{O}(0,2,0,2m),\ \mathcal{O}(0,3,0,2m),\ \mathcal{O}(0,4,0,2m).$$

Theorem 7.7. *Theorems 7.1, 7.2 and 7.5 hold for all algebraically closed field of characteristics not equal to 2.*

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Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803
E-mail address: hongyu@math.lsu.edu

Department of Mathematics and Statistics, Auburn University, Auburn, AL 36849
E-mail address: huanghu@auburn.edu