On the Huynh-Le Quantum Determinant for the Colored Jones Polynomial

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The Colored Jones Polynomial

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- $J'_K(2)$ is the ordinary Jones polynomial for K.
- If K is the unknot $J'_K(N) = 1$.

Determinant Description

Vu Huynh and Thang Le have the following description of the colored Jones polynomial.

Theorem

$$J'_K(N) = q^{(N-1)(\omega(\beta)-m+1)/2}\mathcal{E}_N\left(\frac{1}{\widetilde{det}_q(I - q\rho'(\gamma))}\right)$$

Braids

Let σ_i, $1 \leq i \leq m-1$ be the standard generators of the braid group on m strands.

Definition

For a sequence $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k)$ of pairs $\gamma_j = (i_j, \epsilon_j)$, $1 \leq i_j \leq m-1$ and $\epsilon_j = \pm$, let $\beta = \beta(\gamma)$ be the braid

$$\beta := \sigma_{i_1}^{\epsilon_1} \sigma_{i_2}^{\epsilon_2} \ldots \sigma_{i_k}^{\epsilon_k}.$$

Fix γ so that the closure of $\beta(\gamma)$ is the knot K.
$\omega(\beta)$ is the writhe of $\beta(\gamma)$.

The Operators

Define operators $\hat{x}$ and τ_x and their inverses acting on the ring $\mathcal{R}[x^{\pm 1}, y^{\pm 1}, u^{\pm 1}]$:

$$\hat{x}f(x, y, \ldots) = xf(x, y, \ldots), \qquad \tau_x f(x, y, \ldots) = f(qx, y, \ldots)$$

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Definition

$$a_+ = (\hat{u} - \hat{y}\tau_x^{-1})\tau_y^{-1}, \qquad b_+ = \hat{u}^2, \qquad c_+ = \hat{x}\tau_y^{-2}\tau_u^{-1},$$
$$a_- = (\tau_y - \hat{x}^{-1})\tau_x^{-1}\tau_u, \qquad b_- = \hat{u}^2, \qquad c_- = \hat{y}^{-1}\tau_x^{-1}\tau_u,$$

Quantum Matrices

Definition

A 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is right quantum if

$$ac = qca$$

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An $m \times m$ matrix is right-quantum if all 2×2 submatrices of it are right-quantum.

Quantum Determinants

Definition

If $A = (a_{ij})$ is right-quantum, then the quantum determinant is

$$\det_q(A) := \sum_{\pi \in \mathrm{Sym}(m)} (-q)^{\mathrm{inv}(\pi)} a_{\pi 1,1} a_{\pi 2,2} \dots a_{\pi m,m}$$

where $\mathrm{inv}(\pi)$ denotes the number of inversions.

Quantum Determinants

In general, $I - A$ is no longer right-quantum.

Definition

$$\widetilde{\det}_q(I-A) := 1-C \quad \text{where} \quad C := \sum_{\emptyset \neq J \subset \{1,\dots,m\}} (-1)^{|J|-1} \det_q(A_J),$$

where A_J is the J by J submatrix of A.

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Also

$$\frac{1}{\widetilde{\det}_q(I - A)} = \frac{1}{1 - C} = \sum_{n=0}^{\infty} C^n$$

Operators

Define matrices which are right quantum

$$S_+ := \begin{pmatrix} a_+ & b_+ \\ c_+ & 0 \end{pmatrix} \qquad S_- := \begin{pmatrix} 0 & c_- \\ b_- & a_- \end{pmatrix}$$

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If P is a polynomial operator in the operators $a_\pm$, $b_\pm$, and $c_\pm$ then we get a polynomial $\mathcal{E}_N(P) \in \mathcal{R}$ by having P act on the constant polynomial 1 and replacing x and y with q^{N-1} and replacing u with 1.

The Matrix $\rho'(\gamma)$

Associate to each $\sigma_{i_j}^{\epsilon_j}$ the matrix which is the identity except for the 2×2 minor of rows i_j, $i_j + 1$ and columns i_j, $i_j + 1$ which is replaced by the matrix $S_{\epsilon_j, j}$.

Here $S_{\pm, j}$ is the same as $S_{\pm}$ with x, y, u replaced by x_j, y_j, u_j.

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The matrix $\rho(\gamma)$ is the product of these matrices.
The matrix $\rho'(\gamma)$ is $\rho(\gamma)$ with the first row and column removed.

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Take K to be the right-handed trefoil.
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$$J_K'(N) = q^{N-1}\mathcal{E}_N\left(\frac{1}{1 - qc_1 a_2 b_3}\right) = q^{N-1}\sum_{n=0}^{\infty}\mathcal{E}_N(q^n c_1^n a_2^n b_3^n)$$

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To apply $q^n c_1^n a_2^n b_3^n$ to the constant polynomial 1, recall

$$a_+ = (\hat{u} - \hat{y}\tau_x^{-1})\tau_y^{-1}, \qquad b_+ = \hat{u}^2, \qquad c_+ = \hat{x}\tau_y^{-2}\tau_u^{-1},$$

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So we get

$$q^n x_1^n(u_2 - y_2)(u_2 - q^{-1}y_2)\dots(u_2 - q^{-n+1}y_2)u_3^{2n}$$

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$$q^n x_1^n(u_2 - y_2)(u_2 - q^{-1}y_2)\dots(u_2 - q^{-n+1}y_2)u_3^{2n}$$

$$J'_K(N) = q^{N-1}\sum_{n=0}^{\infty} q^{nN}(1 - q^{N-1})(1 - q^{N-2})\dots(1 - q^{N-n}).$$

Walks

We will define a walk along the braid β from i to j as follows: Begining at the i-th strand, follow the braid along a strand starting at the bottom, until you reach an over crossing. At an over crossing there is a choice to either continue along the strand or jump down to the strand below and continue following along the braid. Continue to the top of the braid ending at the j-th strand.

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Each walk is given a weight defined as follows:
At crossing j:

- If the walk jumps down, assign $a_{\epsilon_j,j}$
- If the walk follows the lower strand, assign $b_{\epsilon_j,j}$
- If the walk follows the upper strand, assign $c_{\epsilon_j,j}$

The weight of the walk is the q times the product of the weights of the crossings.

Picture Page

π-Walks

For $J \subset \{1, \ldots, m\}$ and π a permutation of J, a π-walk is a collection of walks from j to $\pi(j)$ for $j \in J$.

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The weight assigned to a π-walk is the $(-1)^{|J|-1}(-q)^{\mathrm{inv}(\pi)}$ times the product of the weights of the walks in the collection.

More Picture Pages

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Theorem

In Huynh and Le's Theorem

$$J'_K(N) = q^{(N-1)(\omega(\beta)-m+1)/2}\mathcal{E}_N\left(\frac{1}{\widetilde{det}_q(I - q\rho'(\gamma))}\right)$$

$$= q^{(N-1)(\omega(\beta)-m+1)/2}\sum_{n=0}^{\infty}\mathcal{E}_N(C^n)$$

the polynomial C is the sum of the sum of the weights of π-walks for $J \subset \{2, \dots, m\}$.

Sketch of Proof

Part 1:

Claim: The matrix $q\rho(\gamma)$ has entries $a_{i,j} =$ the sum of the weights of the walks from j to i along $\beta(\gamma)$.

Use induction on the length of γ: Obvious for $\gamma = \emptyset$ since $\rho(\gamma)$ is the identity matrix and $\beta(\gamma)$ is the identity in the braid group.

Sketch of Proof

For the inductive step let's look at a specific example:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{+,1} & b_{+,1} & 0 \\ 0 & c_{+,1} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix}$$

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The j-th column becomes:

$$= \begin{pmatrix} a_{1,j} \\ a_{+,1}a_{2,j} + b_{+,1}a_{3,j} \\ c_{+,1}a_{2,j} \\ a_{4,j} \end{pmatrix}$$

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Part 2: Apply $\widetilde{\det}_q(I - q\rho'(\gamma))$

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- Removing 1st row and column corresponds to removing 1 from starting and ending positions.
- $(-1)^{|J|-1}\det_q(A_J)$ is the sum of the weights of the π-walks where π is a permutation of J
- $C = \sum_{\emptyset \neq J \subset \{2,\ldots,m\}} (-1)^{|J|-1}\det_q(A_J)$ is the sum of all π-walks along $\beta(\gamma)$

A Lemma

Lemma

The weights of π-walks which traverse a point on the braid more than once appear in cancelling pairs.

Goal

Conjecture

If K is an alternating knot, then

$$J'_K(N) = \pm \sum_{i=1}^{N} a_{N,i} q^{L_N+i-1} + \ldots \pm \sum_{i=1}^{N} b_{N,i} q^{U_N-i+1}$$

and $a_{N,i} = a_{M,i}$ *for* $i \leq N \leq M$ *and* $b_{N,i} = b_{M,i}$ *for* $i \leq N \leq M$*.*

A Theorem on Positive Braids

Theorem

If K is the closure of the braid β where β has all positive crossings, then

$$J'_K(N) = q^{L_N} + \sum_{i=L_N+N}^{U_N} a_i q^i$$

Moreover, $L_N = (N-1)(\omega(\beta)+m-1)/2$