

Frobenius Extensions, Skein Modules and Landau-Ginzburg Theories

January 23, 2008

In string theory, geometric structures on a phase space are used to construct a Frobenius algebra. The Frobenius algebra in turns gives rise to a topological quantum field theory. In principle the value assigned to surface by the tqft associated to a geometric structure. The dependence on the geometric structure is then eliminated somehow to get a topological invariant.

geometry \rightarrow algebra \rightarrow tqft \rightarrow invariants.

- For the Witten-Reshetikhin-Turaev invariants the moduli of geometric structures is Teichmuller space. The ambiguity is eliminated via a projectively flat connection on the bundle over Teichmuller space where the computations take place.
- For Gromov-Witten theory, it is the space of algebraic maps of curves of genus g with n marked points into a symplectic manifold that represent a particular second homology class. The ambiguity is embraced as defining a family of deformations of the cohomology ring of the manifold.
- For Donaldson invariants, the moduli space is made up of flat self dual connections. He uses degree like arguments using Sard's theorem to prove that the intersection numbers are well defined.

- Heegaard-Floer homology uses a point in Teichmuller space along with an arbitrary desingularization of the induced symplectic structure on the configuration space of choices of n -points to build the TQFT directly. The ambiguity in the construction is eliminated, by taking the homology of the chain complex.

At this point the link homologies due to the approach of Khovanov have deformation spaces, but no attempt has been made to parametrize these deformation spaces in terms of geometric structures on the underlying manifold. In some sense the picture behind link homologies is currently,

Frobenius algebra \rightarrow tqft \rightarrow chain complex

The ambiguity in the construction is eliminated by passing to homology.

The geometric story has yet to be told. In fact, as these theories are only valid in S^3 and cylinders over surfaces.

We are far from having a full physical picture of link homologies.

There is a class of physical theories called Landau-Ginzburg theories that share a lot of structure in common with Link homologies. The point of these lectures will be to build up Landau-Ginzburg theories from a topological viewpoint to the extent we can.

Landau-Ginzburg theories are $3+1$ dimensional TQFTs. They assign a vector space to a triangulated three-manifold and morphisms to triangulated four manifolds. The state spaces are two dimensional foams carried by the triangulation up to equivalence and the morphisms come from averaging over 3-dimensional foams carried by the triangulation of the four manifold. In these theories the triangulation plays the role of geometric input.

So far Uwe Kaiser has a general idea of how to produce the state spaces.

Let R be a ring and let A be a finitely generated free module over R .

We assume that A is a commutative algebra with unit, and the map $u : R \rightarrow A$ given by $r \rightarrow r.1$ is an ring homomorphism. We say that A is a Frobenius extension of R if there is an R -linear map,

$$E : A \rightarrow R$$

and $(x_i, y_i) \in A \times A$ so that for all $a \in A$,

$$a = \sum_i E(ax_i)y_i = \sum_i x_iE(y_ia).$$

We call (x_i, y_i) the dual basis, and we call the pair $E, (x_i, y_i)$ a Frobenius system.

Notice that the Frobenius system gives rise to an isomorphism between A and A^* .

$$\lambda : A \rightarrow A^*$$

given by $\lambda(a)(b) = E(ab)$. The x_i and y_i represent dual basis of A and A^* with respect to this identification hence the appellation *dual basis*.

Alternately we can define $\beta : A \otimes A \rightarrow R$ by $\beta(a, b) = E(ab)$. This is a nondegenerate pairing in the sense that there is a choice of two bases, x_i and y_i so that with respect to these bases the matrix of β is the identity matrix and β is associative in the sense that for all $a, b, c \in A$,

$$\beta(ab, c) = \beta(a, bc).$$

Since A is finite dimensional, multiplication $m : A \otimes A \rightarrow A$ has an adjoint $m^* : A^* \rightarrow A^* \otimes A^*$.

Notice that A^* and $A^* \otimes A^*$ are A -bimodules,

$$a \cdot \phi(b) = \phi(ab) = \phi(ba)(\phi \cdot a)(b),$$

and

$$a \cdot (\phi \otimes \psi) = a \cdot \phi \otimes \psi \neq \phi \otimes \psi \cdot a = (\phi \otimes \psi) \cdot a.$$

Amazingly m^* is a map of bimodules. Also λ being a map of bimodules we can define a bimodule map

$$\Delta : A \rightarrow A \otimes A,$$

by $\lambda^{-1} \otimes \lambda^{-1} \circ m^* \circ \lambda$. This is a coassociative, cocommutative, comultiplication that has the following strange compatibility with the algebra structure,

$$a\Delta(b) = \Delta(ab) = \Delta(a)b.$$

Also $\Delta(1) = \sum_i x_i \otimes y_i$.

Furthermore $E : A \rightarrow R$ is its counit, meaning,

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{Id \otimes E} A \otimes R \cong A$$

and

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{E \otimes Id} R \otimes A \cong A$$

are both the identity map.

Example

Let $A = \mathbb{Z}[a, b][x]/(x^2 - ax - b)$ be an extension of polynomials in a and b with integer coefficients. We grade the algebra and module by letting a and x have degree 2, and b have degree 4. Define $E : A \rightarrow \mathbb{Z}[a, b]$ by $E(x^i) = \delta_1^i$, where we use x^1 , and $x^0 = 1$ as a basis.

All the conditions, the existence of $E, \lambda, \beta, \Delta$ with the properties I gave are equivalent. They could all be given as the definition of Frobenius extension. There are other more elegant definitions of Frobenius extension, and a larger collection of objects that act this way.

From all this structure $End_R(A) \cong A \otimes_R A$ where composition is given by $(a \otimes b)(c \otimes d) = aE(bc) \otimes d = a \otimes E(bc)d$.

The image of 1 under $\Delta, \sum_i x_i \otimes y_i$ acts on $A \otimes_R A$ by $R(a \otimes b) = \sum_i x_i a \otimes y_i b$. Amazingly, this is a solution of the Yang-Baxter equation. The content of Jones tower construction was that he was getting a sequence of solutions of the Yang-Baxter equation by iterating this process, that behaved well with respect to one another.

Example Recall $R = \mathbb{Z}[a, b]$ where we view a as having degree 2 and b having degree 4. We let $A = \mathbb{Z}[a, b][x]/(x^2 - ax - b)$ where we view x as having degree 2. Every coset in A is uniquely represented by a polynomial of the form $\alpha x + \beta$. The Frobenius homomorphism is defined by letting $E(x) = 1$ and $E(1) = 0$ and extending bilinearly from A to R . The Frobenius system is $(x - a, 1), (1, x)$. Being commutative there is a single map $\lambda : A \rightarrow A^*$,

$$\lambda(1) = x^*, \lambda(x) = ax^* + 1^*.$$

The comultiplication is given by

$$\Delta(1) = 1 \otimes X + X \otimes 1 - a1 \otimes 1,$$

and

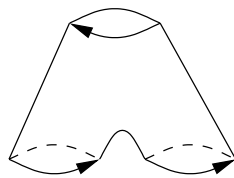
$$\Delta(X) = X \otimes X + b1 \otimes 1.$$

Exercise 1. Let $R = \mathbb{Z}[a, b, c]$ where a has degree 2, b has degree 4 and c has degree 6. Let $A = R[x]/(x^3 - ax^2 - bx - c)$. Let $E(x^2) = -1$, $E(x) = E(1) = 0$. Work out the maps λ , m^* and Δ .

Let R be a commutative ring. A parametrized circle is a circle K along with a homeomorphism $h : S^1 \rightarrow K$ from the standard circle. A collection of n -parametrized ordered circles is n disjoint parametrized circles (h_i, K_i) that have been ordered. Since the notation is unwieldy we will denote a parametrized circle by K and take the parametrization for granted.

If F is an oriented surface with its boundary components parametrized, we can decompose its boundary into the *in* components where the orientation on the component from the parametrization disagrees with the boundary orientation, and the *out* components where the two orientations agree. For the sake of composition we also equip the in and out boundaries of F with an ordering of components.

Below I picture a pair of pants. The in boundary components are down and the out boundary component is up. The boundary component on the bottom left comes before the boundary component on the bottom right. This is the tradition that I am establishing for diagrams. The in boundary on the bottom, the out boundary on the top and the boundary components in the top and bottom ordered from left to right. The orientation on the surface comes from a counterclockwise circle on the part of the surface that is towards you and outside.



The category $R - 2Cobord$ has an object for each nonnegative integer. The morphisms $Mor(m, n)$ is the free R -module on oriented surfaces with ordered parametrized boundary, up to diffeomorphism, so that the in boundary has m components and the out boundary has n components. If m or n is zero, then we mean that the corresponding part of the boundary is empty. By diffeomorphic I mean there is an orientation preserving, order preserving diffeomorphism $h : F \rightarrow F'$ so that when restrict to the i th in or out component of ∂F the composition

$$S^1 \xrightarrow{f_i} F \xrightarrow{h} F' \xrightarrow{f_i'^{-1}} S^1,$$

is the identity map. Where of course $f_i : S^1 \rightarrow F$ and $f_i' : S^1 \rightarrow F'$ are the parametrizations of the respective components.

Composition of morphisms $\circ : Mor(m, n) \times Mor(n, p)$ is required to be bilinear with respect to the R -module structure and on diagrams corresponds to identifying the out boundary of the first morphism with the in boundary of the second morphism via the homeomorphism given by following the parametrizations back to S^1 .

The tensor product of objects is just addition. The tensor product of morphisms is once again bilinear, and on the level of surfaces it is the disjoint union, where we order the boundary components by letting all the boundary components of the second surface follow the first.

There is an adjoint in this category which we denote by $*$. If we change the orientation on the surface F and keep the orderings and parametrizations the same, the in surface becomes the out surface and the out surface becomes the in surface. This give us the linear map

$$* : Mor(m, n) \rightarrow Mor(n, m)$$

which reverses order of composition.

$$(f \circ g)^* = g^* \circ f^*.$$

If A is a commutative Frobenius extension of R , then we can extend this category by having morphisms be R -linear combinations of tuples consisting of an oriented surface with ordered parametrized boundary and an element of A assigned to each component. We might denote one by (F, a) . These tuples are made bilinear by requiring that, for connected surfaces,

$$(F, a) + (F, b) = (F, a + b), r(F, a) = (rF, a) = (F, ra)$$

and for disconnected surfaces the product distributes over addition, so

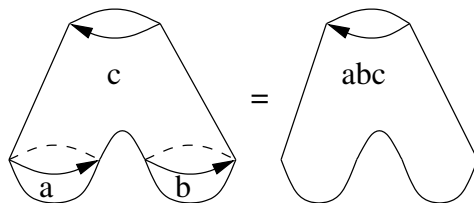
$$(F, a + b) \otimes (G, c) = (F, a) \otimes (G, c) + (F, b) \otimes (G, c),$$

and

$$(G, c) \otimes (F, a + b) = (G, c) \otimes (F, a) + (G, c) \otimes (F, b).$$

To compose two extended surfaces glue them together and multiply all the elements of A that lie on the same component of the glued

together surface. Extend this operation bilinearly.



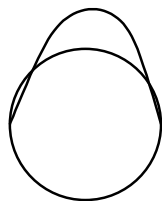
In diagrams a surface without a label is to be thought of as the tuple consisting of the surface and a 1.

Here are some special cobordisms.

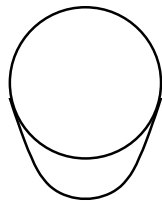
- The identity cobordism is just a cartesian product of circles that have been ordered so that the two ends of any cylinder have the same place in the ordering and the parametrizations correspond under projecting from one end to the other. Each component is decorated with the unit.



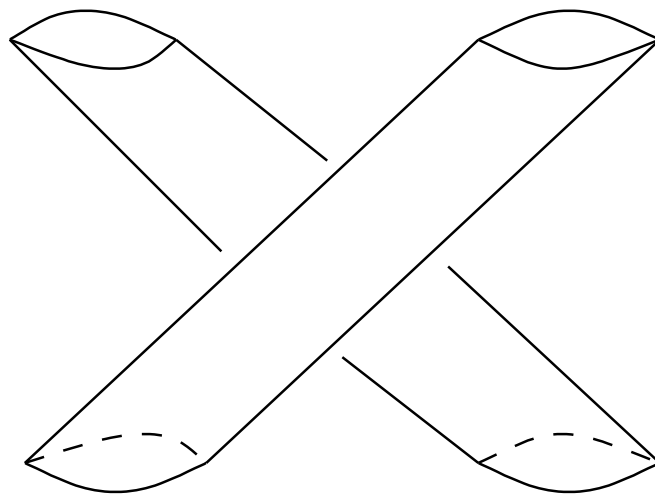
- The Frobenius cobordism is a collection of vertical cylinders except for one disk with a single in component labeled with a 1



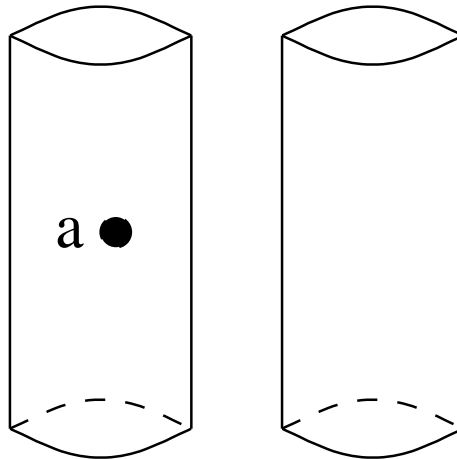
- The unit cobordism is a bunch of cylinders labeled with one and one disk with a single out component.



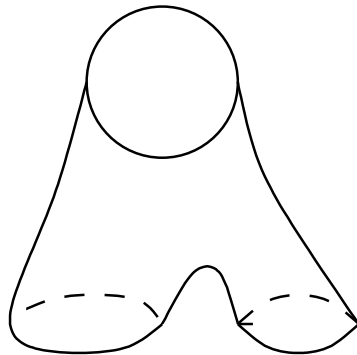
- A permutation cobordism is the same as the identity cobordism except the orderings on the two ends don't agree. There is exactly one permutation cobordism in $Mor(n, n)$ for each permutation in S_n .



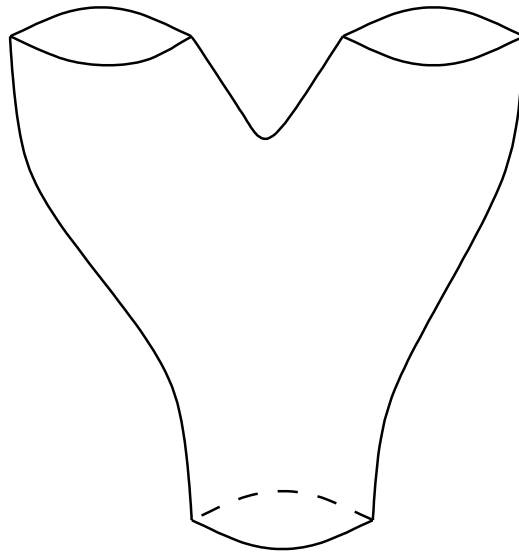
- Multiplication by a in the i th boundary component is a cylinder with the orders matching on the in and out boundary component and the cylinder corresponding to the i th boundary components labeled with an a while all other components carry a 1.



- Multiplying in the i th and $i+1$ th out boundary components. This consists of cylinders over all other boundary components labeled with 1, and a pair of pants whose in boundary is the i th and $i+1$ st boundary components in order, that carries a 1. The ordering of the out boundary components needs to be shifted after the $i+1$ st component in the obvious way.



- Comultiplying in the i th component, is a cylinder over all other components and then a pair of pants labeled with 1 over the i th component. We shift the orderings of in the other out boundary components to compensate for the extra out component.

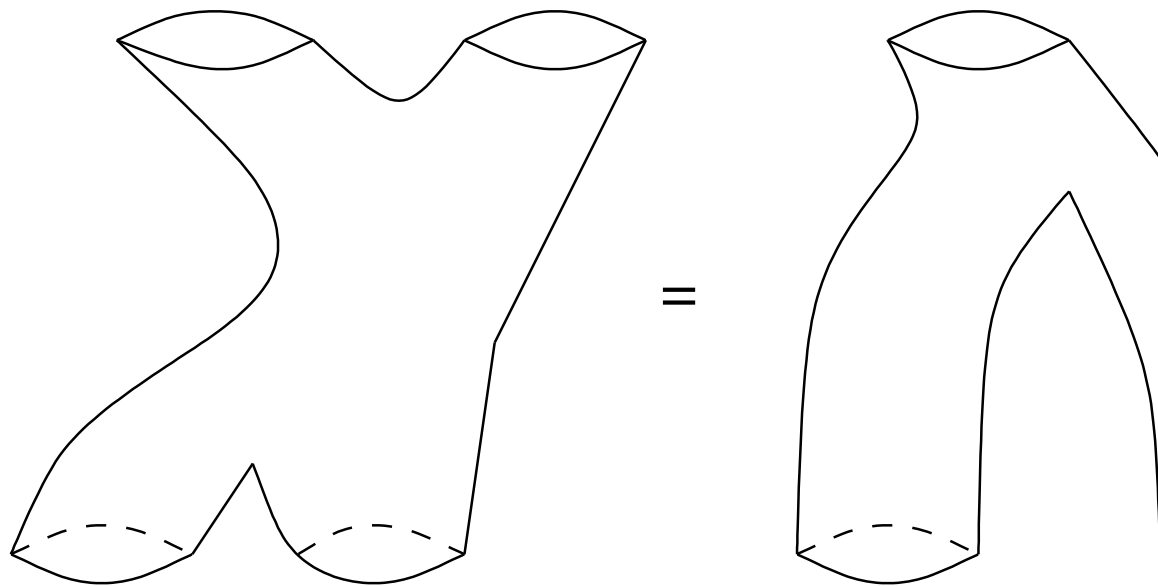


We call this extended category $A/R - 2Cobord$.

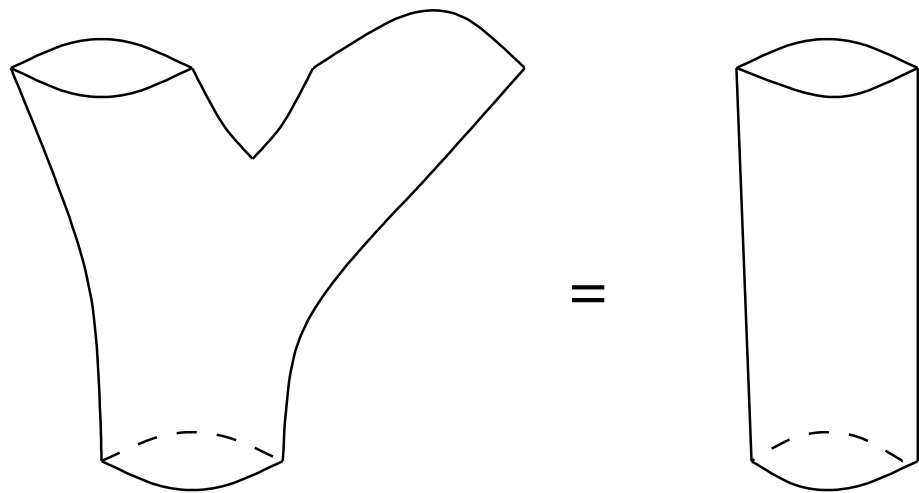
In general we can make Frobenius extensions of R into a category with direct sums, tensor products and morphisms that preserve Frobenius systems. With the appropriate definition of TQFT the category of TQFTs is equivalent to the category of Frobenius extensions. Although it's a beautiful theorem. This notion of extended cobordisms seems more useful for knot theory. The theorem for extended cobordisms would be dumb because I have named the Frobenius extension. As an existential statement the equivalence of these two categories is an amazing and deep theorem that has driven the fields of topology and combinatorial algebraic topology for almost two decades. Unfortunately, it has only very special applications in the course of working examples of link homologies. The concept of extended cobordisms I give here is inspired by the work of Bar-Natan and crystallized by the work of Michael Khovanov and Uwe Kaiser.

Associated to the category $A/R - 2cobord$ is a functor to the following Category. The objects are tensor products of copies of A . We send 0 to R , we send n -circles to the tensor product of n -copies of A . The identity morphism goes to the identity morphism. The unit goes to the morphism inserting an extra copy of A with a 1 in that slot. The Frobenius morphism goes to the morphism that applies the Frobenius mapping E in that coordinate. The permutation bordisms go to the morphism induced by the corresponding permutation in the factors of the tensor product. The multiplication morphism goes to multiplying those two factors together, the comultiplication morphism goes to the morphism where we apply the comultiplication in that entry. If the morphisms are labeled by something other than 1 we also multiply by the element of A that labels it. Since the algebra is commutative these are R -linear maps.

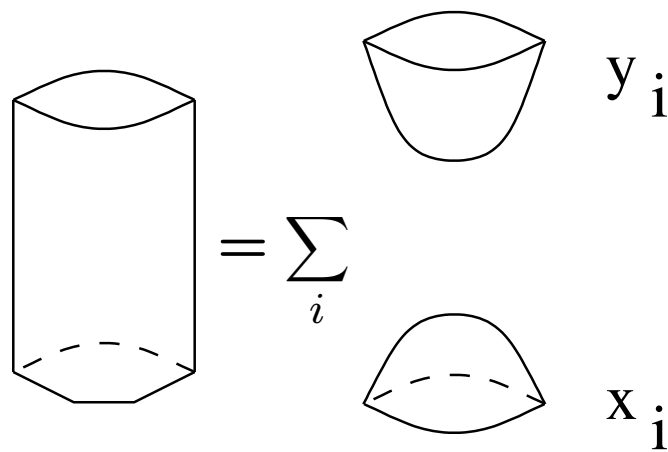
The Frobenius condition implies that the morphism assigned to a marked surface only depends on the marking and the topology of the surface.



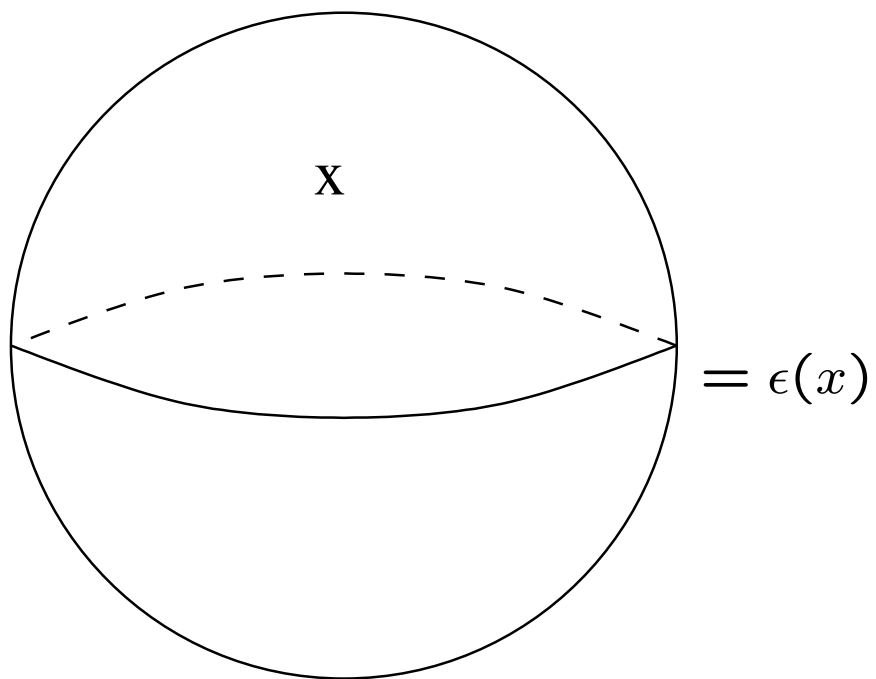
$$a\Delta(b) = \Delta(ab).$$



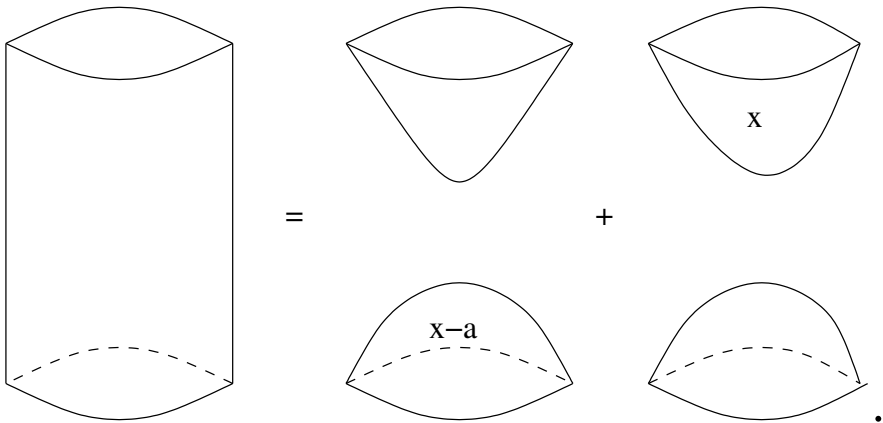
The Frobenius system gives us the following identity.



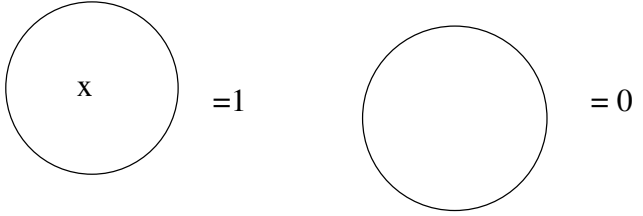
The evaluation of spheres is determined by ϵ .



Going back to our example the neck cutting relation is,



And sphere evaluation is,



Let M be a closed oriented three-manifold. Let R be the base ring. Let \mathcal{F} be the space of all surfaces up to isotopy colored by elements of A . Let $R\mathcal{F}$ be the free module on these. Let S be the submodule that includes the sphere evaluation rules, neck cutting relations, and bilinearity rules for the colorings of the surfaces. We define

$$UK(A)(M) = R\mathcal{F}/S.$$