sl_3-TQFT after Khovanov

January 30, 2008

Any $(1+1)$-dimensional TQFT gives a recipe for assigning a chain complex to a link diagram. (Recall a $(1+1)$-dimensional TQFT is synonymous with an Frobenius algebra A.

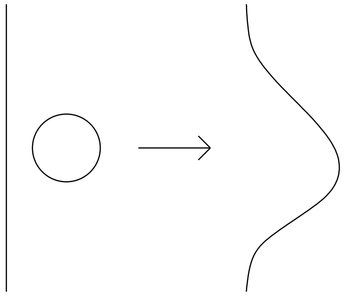
Expand the diagram into states using the Kauffman bracket skein relation. The states are collection of circles which the TQFT sends to tensor product of copies of A.

In the hypercube of states the edges correspond to changing a single minus smoothing to a plus smoothing.

The map we use is a cylinder except for a single saddle where the crossing is changed. This corresponds to either multiplication or comultiplication, depending on whether the change decreases or increases the number of circles in the state.

The hypercube is commutative because TQFT assigns the same morphism to diffeomorphic surfaces. Now concoct a system of signs to make the faces anticommute.

It is rare that the chain complex assigned to a diagram is unchanged up to homotopy by the Reidemeister moves. The big sticking point is the Reidemeister one move.



This gives a chain complex

$$0 \to A \otimes A \to A \to 0,$$

which needs to be homotopic equivalent to

$$0 \to A \to 0.$$

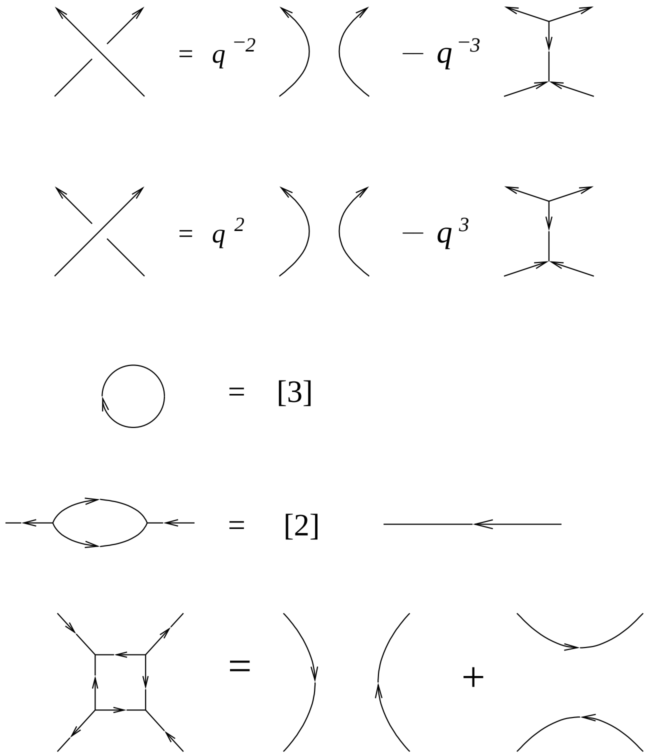
The only way this can happen is if $(dim A)^2 - dim A = dim A$. This implies $dim A = 2$. In fact, the algebra

$$\mathbb{Z}[a,b][x]/(x^2 - ax - b)$$

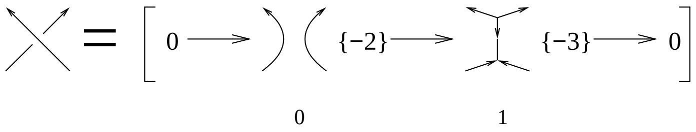
is the most general such algebra and it works. There is a paper by Khovanov where this is worked out.

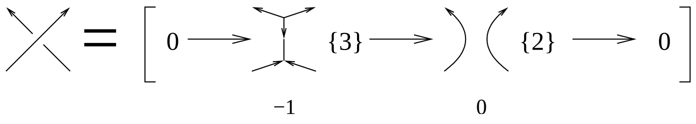
To go farther, we need to use a different skein relation and we need to use a different skein relation. The Homfly skein relation admits *dimer models* where it is computed via states that are trivalent graphs. The first such model was due to Jaeger. This model hasn't been used. Instead there are evaluations of the Homfly polynomial that admit a dimer model due to Murakami, Ohtsuki and Yamada, (MOY) and a special case of this evaluatoin for sl_3 due to Greg Kuperberg that have been used to construct sl_n-theories. This has been done in papers by Mackaay-Paz, Morrison, Stroppel-Mazorchuk, and Kamnitzer, Wu, Khovanov-Rozansky, and Rasmussen.

The skein relations underlying the sl_3-polynomial are, (this diagram is from Khovanov's paper)



The boundary operator corresponding to a resolution comes from plugging in a singular saddle. (The picture is also Khovanov's.)





These theories are implicitly using a broader notion of TQFT that assigns modules to trivalent graphs and morphisms to singular surfaces whose "boundary" is made up of trivalent graphs.

This lecture is provoked by a paper *Network TQFT* by Sergei Natanzon that appeared on the Arxiv in December. It remarks that Khovanov's sl_3-homology theory is based on an example of Network TQFT.

The features of a Network TQFT are

- it assigns a Frobenius algebra to any member of a family of admissibly decorated graphs.
- The foams look locally like cones on admissible graphs.
- You treat an upside down cone as the unit, a right side up cone as the counit.
- There is a relation that comes from doing surgery along a graph that appears as a cross section of a foam.
- The Frobenius interact with one another via disjoint union and saddles.

The goal of this lecture is to explore Natanzon's remark in the special case of the TQFT underlying Khovanov's sl_3-homology.

- Recall, a Frobenius extension $R \to A$ has an R-bilinear map $E : A \to R$ that induces an isomorphism $\lambda : A \to A^*$ by $\lambda(a)(b) = E(ab)$. Of course we are assuming that A is a commutative ring with unit and the inclusion of R into A is by multiples of the unit.

- An excellent source of Frobenius extensions are the cohomology rings of compact oriented manifolds with torsion free homology, and nonzero homology groups only in even dimensions. The Frobenius map is evaluation on the fundamental class.

- Let $\phi : A \to B$ be a linear map of Frobenius extensions. The backwards map

$$\phi^! : B \to A,$$

is constructed from the adjoint of ϕ using the isomorphisms between A and A^* and B and B^*.

- The cohomology ring of the projective plane $\mathbb{C}P(2)$ is

$$\mathbb{Z}[x]/(x^3)$$

with Frobenius mapping,

$$\epsilon : \mathbb{Z}[x]/(x^3) \to \mathbb{Z},$$

given by $\epsilon(x^i) = -1\delta^i_2$. The dual basis is $(1, -x^2), (x, -x), (x^2, -1)$, so the comultiplication is given by

$$\Delta(1) = -1 \otimes x^2 - x \otimes x - x^2 \otimes 1,$$

$$\Delta(x) = -x \otimes x^2 - x^2 \otimes x,$$

$$\Delta(x^2) = -x^2 \otimes x^2.$$

- The complete flags $\mathcal{F}_3$ in $\mathbb{C}$ is the set of filtrations $\{0\} = E_0 \leq E_1 \leq E_2 \leq E_2 = \mathbb{C}^3$ where the quotients E_i/E_{i-1} are all one dimensional. There are three maps $\phi_i : \mathcal{F}_3 \to \mathbb{C}P(2)$ given by $\phi_i(f)$ is the perpendicular to E_{i-1} in E_i. These three maps induce a surjection

$$\Phi^* : H^*(\mathbb{C}P(2))^{\otimes 3} \to H^*(\mathcal{F}_3).$$

- Letting $X = x \otimes 1 \otimes 1$, $Y = 1 \otimes x \otimes 1$ and $Z = 1 \otimes 1 \otimes x$, the kernel of Φ is the ideal generated by the elementary symmetric polynomials,

$$X + Y + Z, XY + YZ + ZX, XYZ.$$

- From these relations we conclude that

$$0 = X(X + Y + Z) = X^2 + XY + XZ$$

Using the second relation above we get,

$$X^2 = YZ.$$

By symmetric we conclude,

$$Y^2 = XZ,$$

and

$$Z^2 = XY.$$

Adding them together, and using that relation again we get,

$$X^2 + Y^2 + Z^2 = 0.$$

Note $0 = Z^4 = X^2Y^2$.

- From these relations we see that X, Y is a a basis for $H^2(\mathcal{F}_3)$, and X^2, Y^2 is a basis for $H^4(\mathcal{F}_3)$. Notice also that

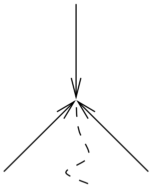
$$XY = Z^2 = -X^2 - Y^2$$

- The dual of the fundamental class of $\mathcal{F}_3$ is then $X^2Y = Y^2Z = Z^2X = -ZX^2 = -XY^2 = -YZ^2$. , and this is a basis for $H^6(\mathcal{F}_3)$.

  The cyclic ordering of the variables makes a difference, in the sign of the fundamental class, other than that the algebra is invariant under the cyclic permutation

$$X \to Y \to Z \to X.$$

  We represent Φ by a trivalent vertex that is cyclically ordered.



- The Frobenius mapping is evaluation on the fundamental class takes a generator of $H^6(\mathcal{F}_3)$ to 1 and all lower order cohomology classes to zero makes $H^*(\mathcal{F}_3)$ into a Frobenius extension of the integers.

- $$\Delta(1) = 1 \otimes X^2Y + X \otimes XY + Y \otimes X^2 +$$
  $$X^2 \otimes Y - Y^2 \otimes X + X^2Y \otimes 1.$$

- In general, if W is a trivalent graph with cyclically oriented vertices, we associate to it the algebra that has one variable for each edge, and whenever three edges, associated to variables X_1, X_2, X_3 meet at a point, we mod out by the relations,

  $$X_1 + X_2 + X_3 = 0,$$

  $$X_1X_2 + X_2X_3 + X_3X_1 = 0,$$

  and

  $$X_1X_2X_3 = 0.$$

  Khovanov also admits some monovalent vertices. If an edge has a monovalent vertex he adds the relation X_i^3. At other edges this is a consequence of the relations. Notice that,

  $$0 = X_1(X_1X_2 + X_2X_3 + X_3X_1) =$$

  $$X_1^2X_2 + X_1X_2X_3 + X_1^2X_3.$$

The middle term is just the last relation, so

$$X_1^2X_2 + X_1^2X_3 = 0.$$

By symmetry we get two more relations like this. From these relations we find,

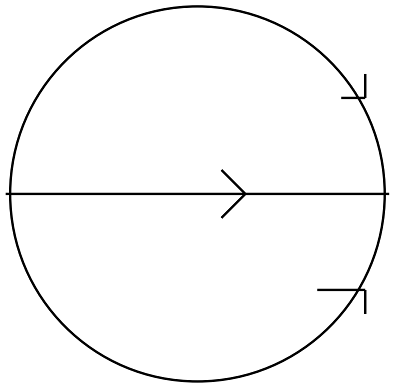
$$X_1^2X_2 = X_2^2X_3 = X_3^2X_1 =$$

$$-X_1^2X_3 = -X_2^2X_1 = X_3^2X_2.$$

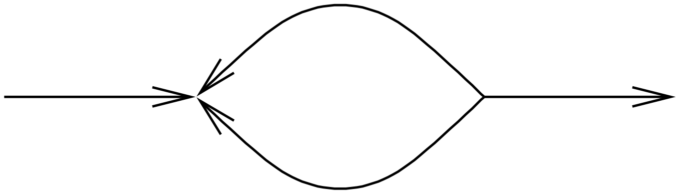
The vertices need to be oriented exactly because of the relation above.

- A more invariant description of the algebra assigned to a graph is take a copy of $H^*(\mathcal{F}_3)$ for each vertex and then tensor them together along one copy of $H^*(\mathbb{C}P(2))$ for each edge.

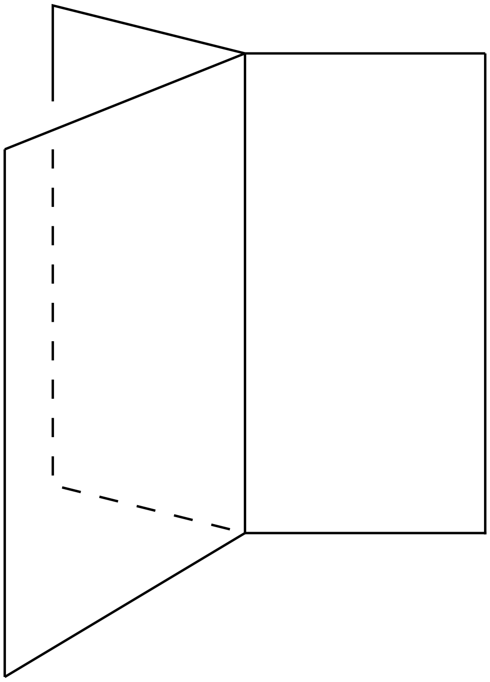
- In the case of two vertices that share three edges, this is just a copy of $H^*(\mathcal{F}_3)$.



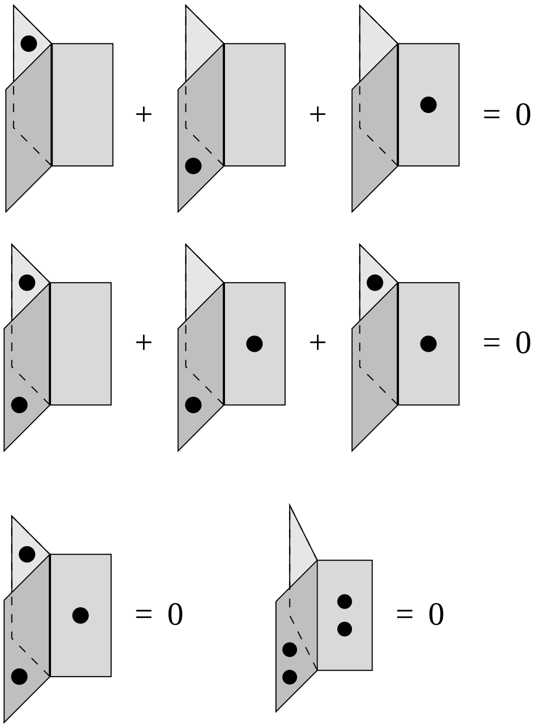
In fact, if they just share two edges, its enough the algebra is still just $H^*(\mathcal{F}_3)$.



A foam is a singular surface where the singularities are modeled on the letter Y cross an interval, and there is a cyclic orientation along each edge.



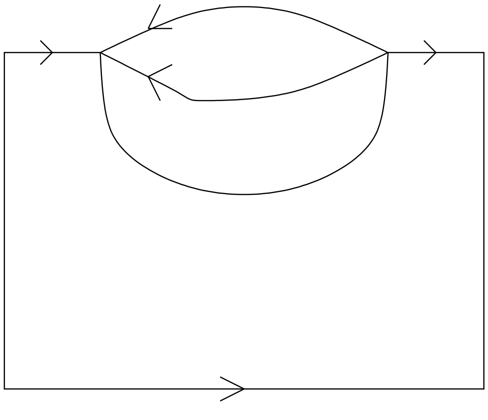
The following identities capture the relations between the generators at a vertex. I lifted the figure below from his paper.



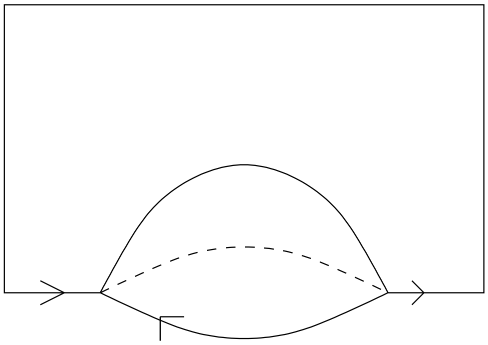
A diagram represents a monomial with one power of each variable for each dot on the corresponding face. The top row is $X+Y+Z=0$.

If the diagram is just a vertical Y without dots it should be the identity map. Putting dots on, corresponds to multiplying by the appropriate variable.

The figure below should correspond to some map from the cohomology ring of $H^*(\mathbb{C}P(2))$ to $H^*(\mathcal{F}_3)$. We can think in formulas of sending 1 to 1, X to X and X^2 to X^2.



The figure below should be the reverse mapping



In formulas using X^2Y as the fundamental class and letting the big sheet towards us be X, the front of the bubble be Y and the back of the bubble be Z,

$$1, X, X^2 \to 0$$

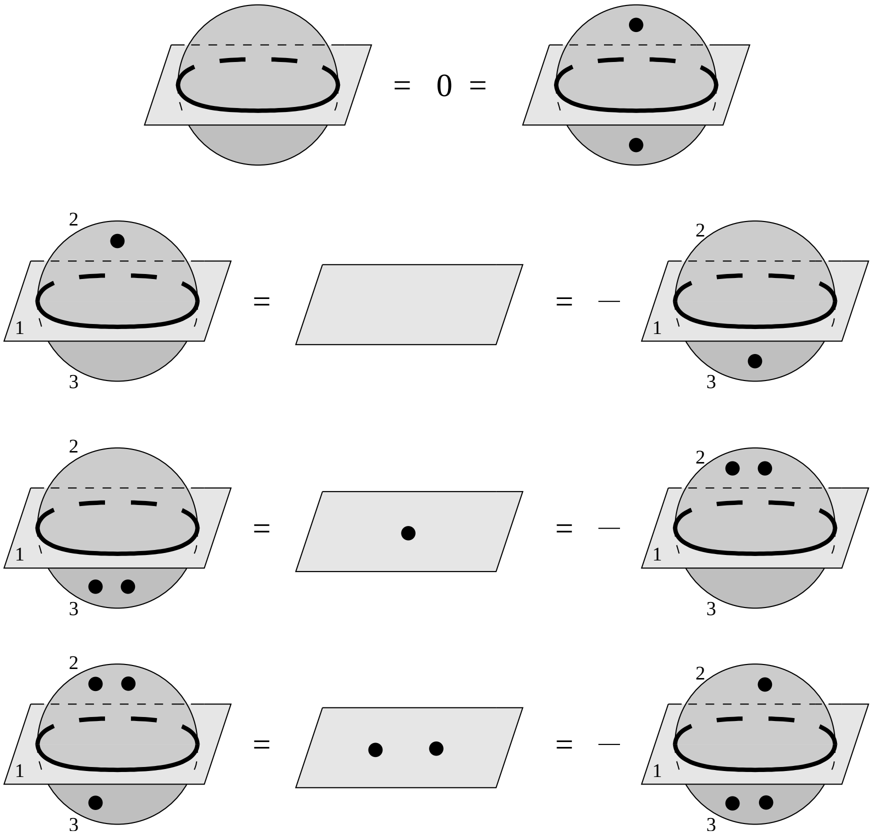
$$YX^2 \to X^2$$

$$YX \to X$$

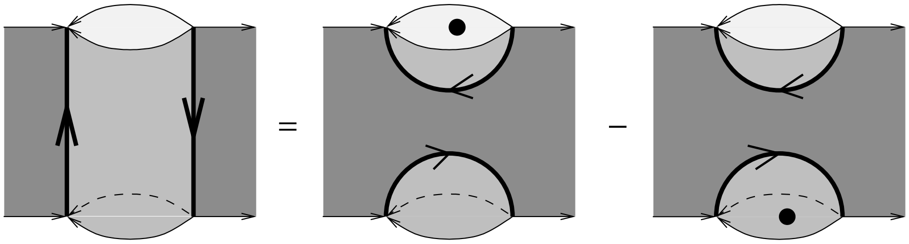
$$Y \to 1.$$

Notice $XY = -X^2 - Y^2$, so Y^2 is sent to X. Notice $YZ = X^2$ so YZ is sent to zero.

A good test is to look at Khovanov's bubble popping identities, which are also heisted from his paper.

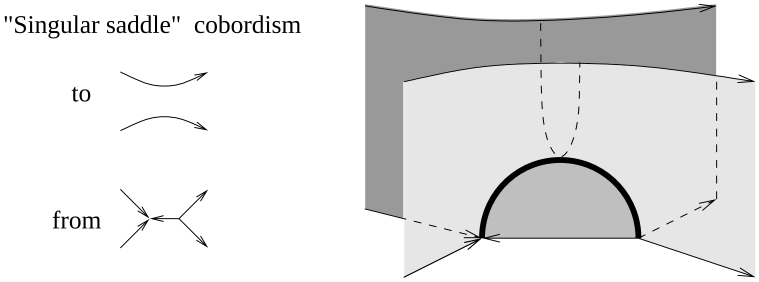


Looking at the formulas his identity,



is just the composition of the two maps I proposed.

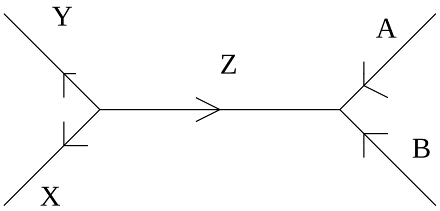
There is one more figure that I need to associate a morphism to.



Khovanov elegently avoids working out the full properties of this map, so the reverse engineering is a little more difficult.

A couple more days on this lecture and the next bit would be a lot more solid.

To begin with the algebra associated to the base is generated by X, Y, Z, A, B



We have relations

$$X + Y + Z = 0, A + B + Z = 0,$$

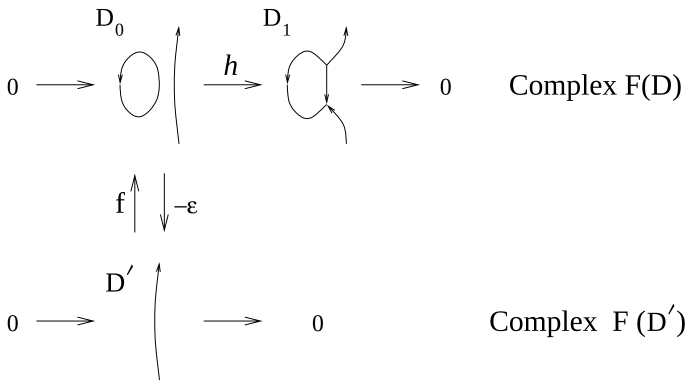
$$XY + YZ + ZX = 0, AB + BZ + ZA = 0,$$

$$XYZ = 0, ABZ = 0, X^3 = Y^3 = A^3 = B^3 = 0.$$

I believe this ring is a rank 12 free module over the integers with 1 generator in dimensions 0 and 8. Three generators in dimensions 2 and 6 and four generators in dimension 4. It satisfies Poincare duality, so it is a Frobenius algebra and the Frobenius maps sends anything equivalent to A^2X^2 to one, and all lower dimensional classes to 0.

The algebra of the two arcs has dimension 9. The map corresponding to the singular saddle is onto and has kernel of dimensions 3. Say the two edges on the top have generators C and D. The maps sends X, B to C, and A and Y to D. It seems that every other value is forced by these choices. The upside down saddle is the reverse mapping.

The obstruction to having a dimer model assign chain complexes that are unchanged up to homotopy by the Reidemeister moves still seems to be the Reidemeister one move that gives a comparison chain complex like this.



Notice the dimensions work out!

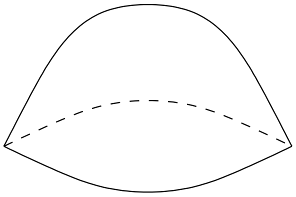
Towards checking we have a TQFT, we need to associate a morphism to each foam. To this end lets think about what a Morse function on a foam would look like.

The idea is to think of the singular curves as smooth.

There are only go to be local minimums and local maximums along singular curves. However, generically we can have two fins up or two fins down. These are the figures I obsessed on this whole lecture. We assign these objects

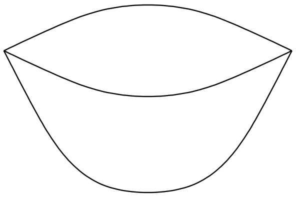
In addition we have the standard Morse critical points away from the singluar set.

These are local maximums,

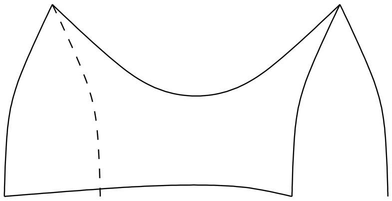


To which we assign the Frobenius mapping for $H^*(\mathbb{C}P(2))$.

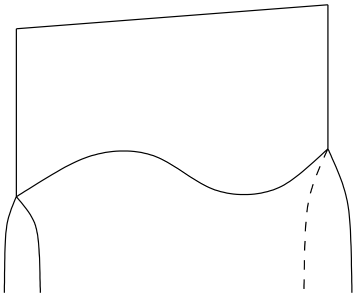
Local minimums to which we assign the unit.



And saddles that get the multiplication or comultiplication depending on whether they connect components or break components apart.



To finish these we need to understand the codimension one singularties in the space of Morse functions on foams and see that the morphism built from these pieces is invariant under them.



Above is the picture of a the result of a $0-1$-birth along an edge with two sheets down and one sheet up. There is a similar birth along a singular edge with one sheet down. We might need some moves where you tip a singular arc, and the number of sheets that are up or down change.

A consequence of all this is that the vector space assigned to a graph is a Frobenius algebra. This means that you have a unit, a counit and a dual basis. We probably want to add vertices to Khovanov's theory so we can represent the counit and unit as foams that are a cone on the graph. There will be a surgery relation for cutting along a graph forming two cones.

Mackaay and Paz construct invariants based on the most general three dimensional algebra,

$$\mathbb{Z}[a,b,c][x]/(x^3 - ax^x - bx - c).$$

To set up TQFTs underlying sl_n we have to color the edges of our dimer models. It is just a pun that for sl_3, $\Lambda^{(3-1)}V \cong V$, for V the fundamental three dimensional representation. Also, you need to admit singularities in the foams corresponding to a vertex of a trivalent spine.

Following Uwe's method, it seems clear how to construct state spaces associated to three manifolds based on foams. You need to include more relations than for surfaces because you need to have relations that indicate the relations between the morphisms assigned to nonsurface critical points.

The really big mystery is how to you associate a morphism between state spaces of three-manifolds from a four-manifold.