State-Sum Models for the HOMFLY Polynomial

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We will begin by defining $H_K(z, a)$, a polynomial of the link K, using skein relations. We will then normalize H_K to obtain P_K, the HOMFLY polynomial of K.

1. If the links K and K' are regularly isotopic (Reid. II, III moves) then $H_K(z, a) = H_{K'}(z, a)$.
2. $H_{unknot}(z, a) = 1$
3. $H_{(+xing)}(z, a) - H_{(-xing)}(z, a) = zH_{(sm)}(z, a)$
4. $H_{(+kink)}(z, a) = aH_{(sm)}(z, a)$ and $H_{(-kink)}(z, a) = a^{-1}H_{(sm)}(z, a)$

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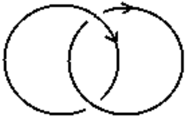
To normalize $H_K(z, a)$ we must use $w(K)$, the writhe of K. Then the HOMFLY polynomial, $P_K(z, a)$, is defined:

$$P_K(z, a) = a^{-w(K)} H_K(z, a).$$

The skein model I just described gives us the following HOMFLY polynomials:



For the Hopf Link:

$H_{Hopf}(z, a) = za + (a - a^{-1})z^{-1}$

$P_{Hopf}(z, a) =$
$za^{-1} + z^{-1}a^{-1} - a^{-3}z^{-1}$

For the (positive) Trefoil:

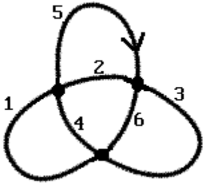
$H_{tref}(z, a) = z^2 a + 2a - a^{-1}$

$P_{tref}(z, a) = z^2 a^{-2} + 2a^{-2} - a^{-4}$

In the model developed by Jaeger and expanded on by Kauffman (in *Combinatorics and topology - Francois Jaeger's work in knot theory*), we turn our oriented link diagram into a 4-valent graph, then use this graph to compute the HOMFLY polynomial.

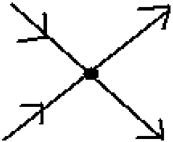
We begin with our original oriented link diagram K, we change all crossings to vertices of 4-valent graphs and label the resulting edges with integers $\{1, 2, \ldots, m\}$ with respect to orientation. This ordering will be something we'll need later on.

BECOMES

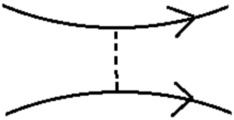


Given our link diagram K, we'll create a state, S, which is a 4-valent graph, by choosing one of two options at each crossing (respecting original orientation, of course). Note that if the diagram originally has n crossings, there will be 2^n possible states.

Either:



splitting



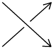
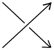
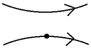
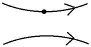
OR smoothing

For each state S, we take a *walk* on S. We always start at the lowest-labeled edge, and when we encounter a crossing, we go through the vertex (i.e. to the edge that is *not* adjacent to the current edge). As we walk we keep track of two specific things:

- The first time we pass through a splitting, we turn our current path into the overcrossing.
- The first time we pass through a smoothing, we mark the passage with a dot.

If we complete an entire cycle, and still have unused edges, we repeat this process starting at the lowest-labeled unused edge.

Once all states have been walked, it is time to calculate the HOMFLY! Each state S will contribute a state value, $< K|S >$, to the overall polynomial. We find $< K|S >$ by multiplying the vertex weights given below for all vertices in the state.

-  $= a$
-  $= a^{-1}$
-  $= \begin{cases} z & \text{if original crossing in K was positive} \\ 0 & \text{otherwise} \end{cases}$
-  $= \begin{cases} -z & \text{if original crossing in K was negative} \\ 0 & \text{otherwise} \end{cases}$

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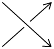
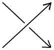
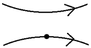
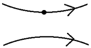
-  $= a$
-  $= a^{-1}$
-  $= \begin{cases} z & \text{if original crossing in K was positive} \\ 0 & \text{otherwise} \end{cases}$
-  $= \begin{cases} -z & \text{if original crossing in K was negative} \\ 0 & \text{otherwise} \end{cases}$

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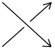
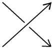
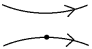
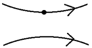
-  $= a$
-  $= a^{-1}$
-  $= \begin{cases} z & \text{if original crossing in K was positive} \\ 0 & \text{otherwise} \end{cases}$
-  $= \begin{cases} -z & \text{if original crossing in K was negative} \\ 0 & \text{otherwise} \end{cases}$

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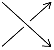
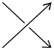
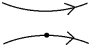
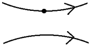
-  $= a$
-  $= a^{-1}$
-  $= \begin{cases} z & \text{if original crossing in K was positive} \\ 0 & \text{otherwise} \end{cases}$
-  $= \begin{cases} -z & \text{if original crossing in K was negative} \\ 0 & \text{otherwise} \end{cases}$

Then, we let

$$H_K(z,a) = \sum_S < K|S > \delta^{||S||-1}$$

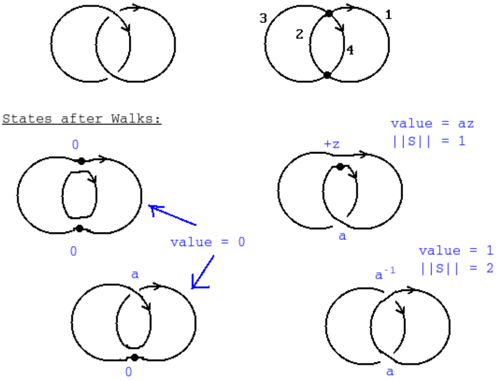
where

- $||S||$ = the number of components of S.
- $\delta = (a - a^{-1})z^{-1}$ (the loop value for the HOMFLY)

Finally, as in the original setup

$$P_K(z,a) = a^{-w(K)} H_K(z,a)$$

The Hopf Link, using Jaeger-Kauffman model



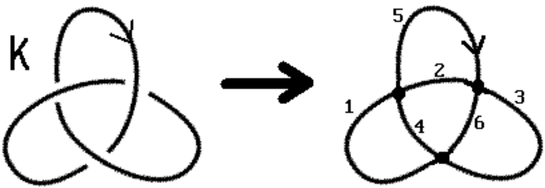
Giving:

$H_K(z, a) = za + 1(a - a^{-1})z^{-1}$

$P_K(z, a) = za^{-1} + z^{-1}a^{-1} - a^{-3}z^{-1}$

The state-sum method we just saw is related to the idea of **skein trees**, which are introduced in another paper of Kauffman's, *State models for link polynomials*.

Skein trees begin with the same idea of turning our link diagram K into a 4-valent graph and labeling the edges with integers.



Instead of looking at all possible states, we use a tree idea to hone in on only the states which turn out to be meaningful. The idea is to travel around our diagram K, and at each crossing, only make changes if they make the knot closer to being ascending (and thus closer to being an unlink).

Like before, we begin at the lowest labeled edge in K.

We travel around K in the direction of orientation until we come to an **undercrossing**. This becomes our active crossing, and two branches (say K' and K'') are added to the tree extending down from the node K. One branch changes the undercrossing to an overcrossing, and the other changes the undercrossing to a smoothing, with a dot marking the side the path will continue on.

We continue this process with all new nodes. A node becomes an **end-node** and contributes to the HOMFLY polynomial once we are able to travel from our starting edge and go through every crossing as an overcrossing first.

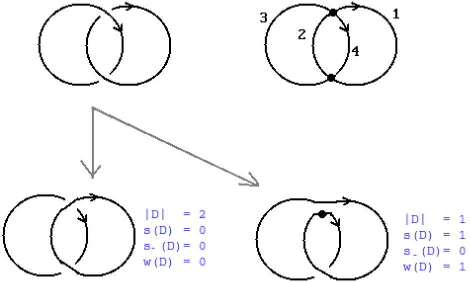
For each end-node diagram D, let:

- $|D|$ = the number of components in D
- $s(D)$ = the total number of smoothings taken to obtain D from K
- $s\_(D)$ = the number of smoothings of NEGATIVE crossings taken to obtain D from K
- $w(D)$ = writhe of D
- $\delta = (a - a^{-1})z^{-1}$

Then the HOMFLY polynomial of K is

$$P_K(z,a) = a^{-w(K)} \sum_D (-1)^{s_-(D)} z^{s(D)} a^{w(D)} \delta^{|D|-1}$$

The Hopf Link, using skein tree



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In *HOMFLY Polynomial via an Invariant of Colored Plane Graphs*, Murakami, Ohtsuki, and Yamada used a combinatorial idea to construct an invariant of colored, oriented, trivalent plane graphs. They go on to define a one-variable version of the HOMFLY polynomial in terms of this invariant. In a later paper, Rasmussen expanded the MOY setup to define the polynomial over two variables.

To begin the MOY process, we fix an integer $n \geq 2$. (We will get an $sl(n)$ invariant depending on our choice of n)

Given any trivalent graph G, we can orient its edges (at each vertex, we always have two in and one out or vice versa) and label them with positive integers k. Through a process described in their paper, we use the orientations and labelings to turn G into a union of simple closed curves, each of which is labeled with k elements of $N = \{\frac{-(n-1)}{2}, \frac{-(n-1)}{2} + 1, \ldots, \frac{(n-1)}{2}\}$.

The eventual bracket formula we get looks like

$$< G >_n = \sum_{labelings:\sigma} \{\prod_{vertices:v} wt(v;\sigma)\} q^{rot(\sigma)}$$

The process of taking this bracket invariant for even a simple graph quickly becomes very involved!

However, MOY go on to prove some lemmas which turn out to be extremely helpful in doing actual computations and eventually working with the HOMFLY polynomial of links.

In these lemmas they use quantized integers, defined as below:

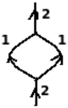
$$[k] = \frac{q^{k/2} - q^{-k/2}}{q^{1/2} - q^{-1/2}} = \sum_{i \in N} q^i$$

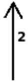
Lemma 1

$\Big\langle$ 

$\Big\rangle_n = [n] < \emptyset >_n$

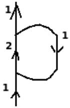
Lemma 2

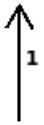
$\Big\langle$ 

$\Big\rangle_n = [2] \Big\langle$ 

$\Big\rangle_n$

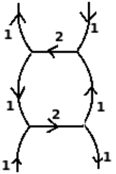
Lemma 3

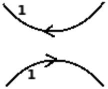
$\Big\langle$ 

$\Big\rangle_n = [n-1] \Big\langle$ 

$\Big\rangle_n$

Lemma 4

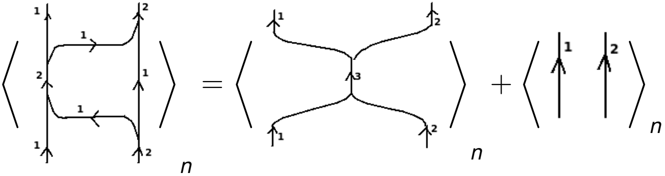
$\langle$ 

$\rangle_n = [n-2] \langle$ 

$\rangle_n + \langle$ 

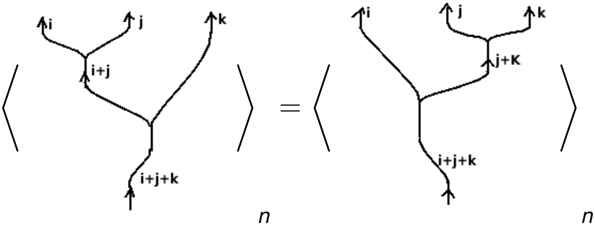
$\rangle_n$

Lemma 5

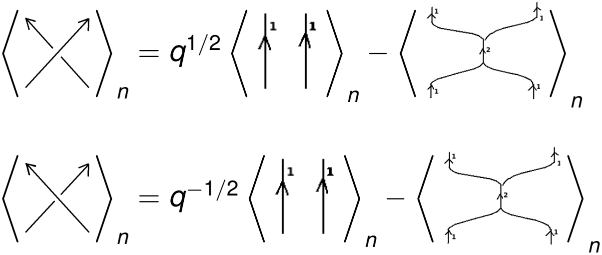


Lemma 6

For any i, j, k positive integers



We now move on to calculation of the HOMFLY polynomial using the bracket defined on trivalent graphs. The local relations which let us take $< D >_n$ of a link diagram D are:



This recursive definition of our bracket on link diagrams is proven to be invariant under Reidemeister II and III. Just as in all the previous setups, we normalize this bracket to get invariance under Reidemeister I.

$$P_n(D) = q^{(n/2)(-w(D))} < D >_n$$

Which is a version of the HOMFLY polynomial.

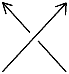
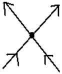
This can be expressed as a state-sum as well, but it is more complicated than I have time for here.

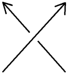
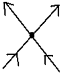
Jacob Rasmussen, in *Some Differentials on Khovanov-Rozansky Homology*, devotes a section to discussion and expansion of the ideas presented in the MOY paper.

He returns to the idea of 4-valent graphs, and as in our first model, creates a state σ and 4-valent graph D_σ by resolving each crossing in the link diagram D to either a splitting or a smoothing.

Each resolution of a crossing has a *local weight* assigned as follows:

All smoothings equal zero.

 to  $= 1$

 to  $= -1$

The local weights in a state σ sum to give the *state weight*, $\mu(\sigma)$

The unnormalized HOMFLY polynomial is then given as:

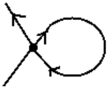
$$\tilde{P}(L) = (aq^{-1})^{w(D)} \sum_{states:\sigma} (-q)^{\mu(\sigma)} \tilde{P}(D_\sigma)$$

Where $\tilde{P}(D_\sigma)$ is calculated using some lemmas, as in MOY.

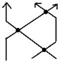
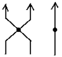
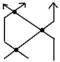
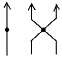
BIG NOTE:
Rasmussen's q = MOY's $q^{-1/2}$
Rasmussen's a = MOY's $q^{-n/2}$

Using these equalities, the MOY and Rasmussen setups give the same invariant, but we see that Rasmussen has freed a from any attachment to q.

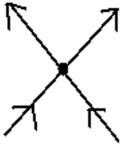
- **Lemma 0:** $\tilde{P}\Bigg($  $\Bigg) = \frac{a-a^{-1}}{q-q^{-1}}\,\tilde{P}\Bigg($  $\Bigg)$

- **Lemma 1:** $\tilde{P}\Bigg($  $\Bigg) = \frac{aq^{-1}-a^{-1}q}{q-q^{-1}}\,\tilde{P}\Bigg($  $\Bigg)$

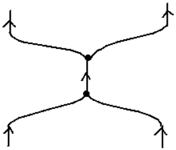
- **Lemma 2:** $\tilde{P}\Big($  $\Big) = (q + q^{-1})\,\tilde{P}\Big($  $\Big)$

- **Lemma 3:**
  $\tilde{P}\Big($  $\Big) + \tilde{P}\Big($  $\Big) = \tilde{P}\Big($  $\Big) + \tilde{P}\Big($  $\Big)$

It is not immediately obvious, but Rasmussen's setup indeed gives another way to calculate the MOY invariant - in fact, the MOY invariant is a specialization of the Rasmussen invariant where $a = q^n$. To see how the lemmas in the two different setups relate, we can think of stretching the vertices from the 4-valent graphs, so that they become edges:



becomes



- My goal in working through these specific models is to gain a thorough understanding of the HOMFLY polynomial for links, and to be able to view it in a variety of different ways.
- A longer-term goal is to apply this understanding to categorification of the HOMFLY polynomial, much like what we have seen with the categorification of the Jones polynomial throughout this semester.

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