Knot Floer Homology for some fibered knots

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Knot Floer homology review

Given

- Y, a compact oriented 3-manifold
- $K \subset Y$, an oriented knot
- $S \subset Y$, an compact, oriented surface, properly embedded in Y, with $\partial S = K$

P. Ozsváth and Z. Szabó construct abelian groups

$$i \in \mathbb{Z} \longrightarrow \widehat{HFK}(Y, K; i)$$

that are isotopy invariants of K. These are called the knot Floer homology groups of K.

Knot Floer review – cont.

For $\widehat{HFK}(Y, K; i)$ we know

- There is a $\mathbb{Z}/2\mathbb{Z}$-grading
- Let $\Delta_{Y,S}(T)$ be the Alexander polynomial determined by S. Then

$$\Delta_{Y,S}(T) \doteq \sum_i a_i T^i$$

where

$$\chi(\widehat{HFK}(Y, K; i)) = a_i$$

in the $\mathbb{Z}/2\mathbb{Z}$-grading.

Knot Floer review – cont.

For Y a $\mathbb{Q}HS$, the $\mathbb{Z}/2\mathbb{Z}$ grading can be lifted to a rational grading. We know that

$$\widehat{HFK}(Y, K; i) \cong \bigoplus_{\mathfrak{s}\in \mathrm{Spin}^c(Y)} \widehat{HFK}(Y, K; \mathfrak{s}, i)$$

and for each $\mathfrak{s} \in \mathrm{Spin}^c(Y)$ there is a number $d(\mathfrak{s}) \in \mathbb{Q}$ such that

$$\xi \in \bigoplus_{i\in\mathbb{Z}} \widehat{CFK}(Y, K; \mathfrak{s}, i) \Rightarrow gr_{\mathbb{Q}}(\xi) - d(\mathfrak{s}) \in \mathbb{Z}$$

whose parity is the $\mathbb{Z}/2\mathbb{Z}$-grading.

Knot Floer review – final

Furthermore, when Y is $\mathbb{Q}HS$, $\Delta_{Y,S}(T)$ can be normalized by requiring

- $\Delta_{Y,S}(T^{-1}) = \Delta_{Y,S}(T)$
- $\Delta_{Y,S}(1) = |H_1(Y, \mathbb{Z})|$

The three manifolds

Y will be obtained as a branched double cover of S^3, branched over a link L. We will denote this $\Sigma(L)$. When L is non-split alternating

$$|H_1(\Sigma(L), \mathbb{Z})| = \det(L) < \infty$$

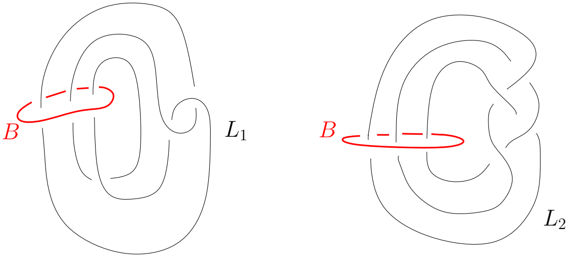
The knots

$K \subset \Sigma(L)$ will be the double cover of an unknot in S^3. In particular, assume

- L lies in $S^1 \times D^2 \cong \{1 < |z| < 2\} \times I$ where the annulus is in the xy-plane.
- $|L \cap D^2| = 2g + 1$.

K is then the double cover of $B = \partial D^2$. S is the branched double cover of D^2, a genus g surface.

Two examples

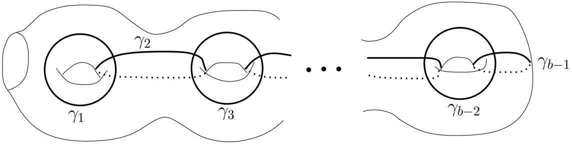


The first has branched double cover S^3, in which the lift of the red unknot is the knot 6_1. The right braid closure is the figure eight knot, and has double branched cover $L(5, 2)$.

Fibered knots

When $\mathbb{L}$ is a braid, Each $\{p\} \times D^2$ will lift to a genus g surface, and can be extended to a fiber of a open book with binding K.

In particular, postive crossings of the braid L correspond to right hand Dehn twists on S, and negative crossings correspond to left hand Dehn twists. The correspondence is given by taking generator σ_i to D_{γ_i} where



By a theorem of Penner, we can obtain pseudo-Anosov mapping classes by taking monodromies of the following form:

$$D_{\gamma_{i_1}}^{n_1} \cdots D_{\gamma_{i_k}}^{n_k}$$

where $\gamma_{i_l} \cap \gamma_{i_j} \neq 0$ implies $\mathrm{sgn}(n_l) = -\mathrm{sgn}(n_j)$, and $\{i_1, \ldots, i_k\} = \{1, \ldots, 2g\}$.

This corresponds to L being an alternating, non-split braid closure.

Two theorems

Theorem (L.R.)

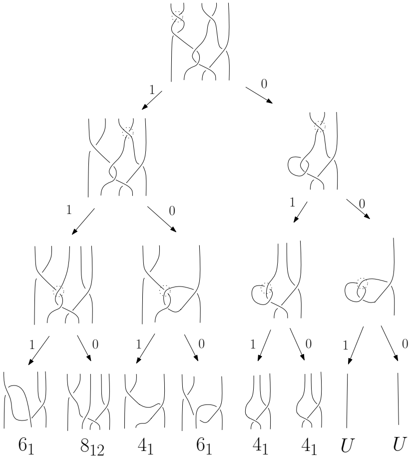
Let $\mathbb{L}$ be a non-split, alternating link in $A \times I$, with $\det(\mathbb{L}) \neq 0$, and which intersects the spanning disc for B in an odd number of points. Then the $\mathbb{Z}/2\mathbb{Z}$-grading of every element of $\widehat{HFK}(\Sigma(\mathbb{L}), K; i, \mathbb{F}_2)$ is determined by the parity of i. Thus the dimension of the homology for each i is determined by the coefficient of T^i in $\Delta_{Y,S}(T)$.

Theorem (L.R.)

Let $\mathbb{L}'$ be $\mathbb{L}$ with two copies of the center of the annulus, unlinked from $\mathbb{L}$. There is a spectral sequence whose E^2 term is isomorphic to the reduced Khovanov skein homology of the mirror, $\overline{\mathbb{L}}'$, in $A \times I$, with coefficients in $\mathbb{F}_2$ and which converges to $\oplus_{i \in \mathbb{Z}} \widehat{HFK}(\Sigma(\mathbb{L}) \#^2 (S^1 \times S^2), \widetilde{B} \# B(0,0), i, \mathbb{F}_2)$.

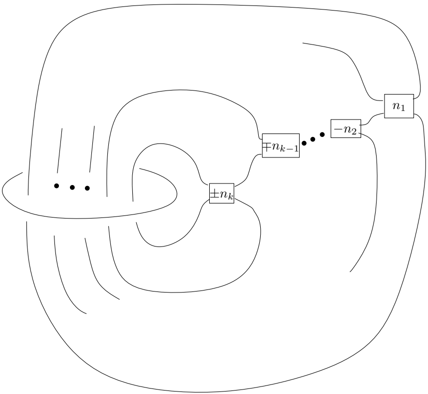
These two theorems generalize to the knot Floer homology setting results of P Ozsváth and Z. Szabó concerning $\widehat{HF}(\Sigma(L))$.

Proving Theorem I:



Start with the closure of an alternating tangle. Pick a crossing and resolve the crossing in both possible ways. Do this only if the result of each resolution is still non-split (it will be alternating). If one or other is not, move on to another crossing. At the leaves of this tree will be twisted unknots. The lift of B in S^3 for each is listed underneath.

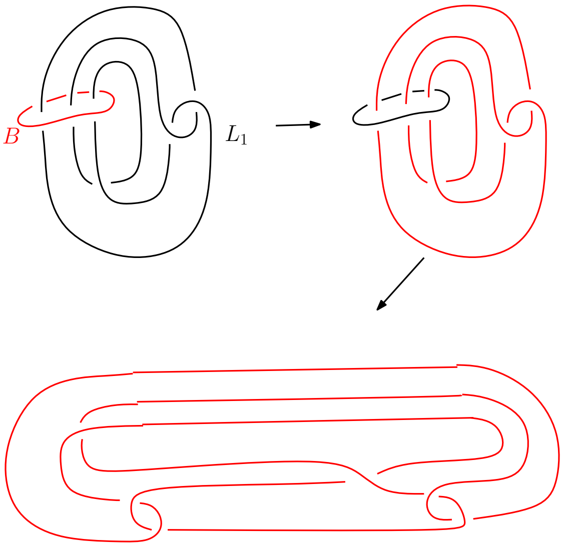
Proving Theorem I - cont.



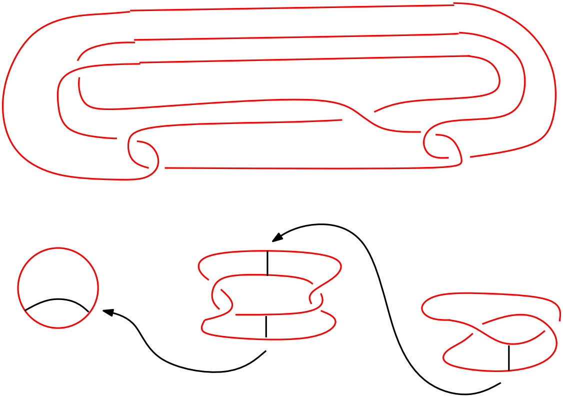
The leaves are as on the right. We record a few facts about K for these branch loci.

- $\bullet K \subset S^3$
- $\bullet K$ is alternating (the link is symmetric).
- $\bullet K$ is a plumbing of twisted bands

For example,



As a plumbing:



More results for twisted unknots

Since K is such a plumbing, we can compute its signature $\sigma(K)$ as the signature of

$$\begin{bmatrix} 2t_1 & 1 & \cdots & 0 \\ 1 & 2t_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2t_k \end{bmatrix}$$

where $t_i = \pm n_i$, depending upon the twisting. A direct computation gives $\sigma(K) = \sum_{i=1}^{k} \mathrm{sgn}(t_i) = 0$.

By a theorem of P. Ozsváth and Z. Szabó, an alternating knot in S^3 has the property that

$$\widehat{HFK}(S^3, K, i) \cong \mathbb{Z}^{|a_i|}$$

in $\mathbb{Q}$-grading $i + \frac{\sigma(K)}{2}$. For K found from a twisted unknot this means

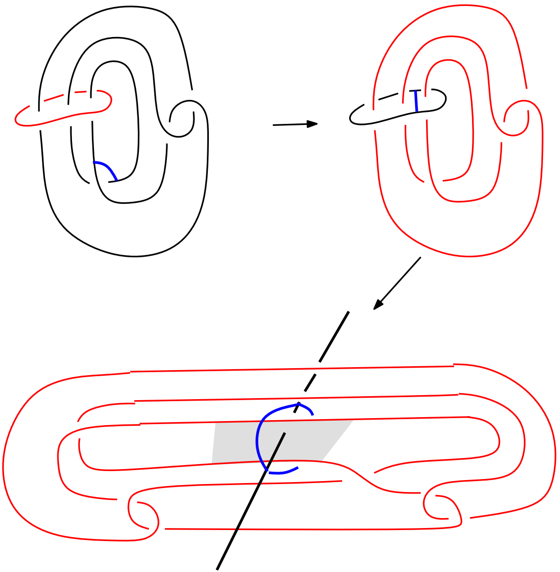
- The Alexander-Conway polynomial determines the homology
- The $\mathbb{Z}/2\mathbb{Z}$-grading is given by $i \bmod 2$.
- $\tau(K) = 0$.

Completing the argument

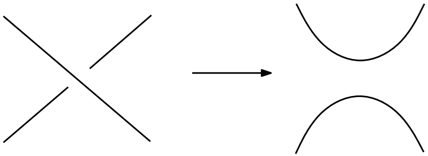
Proceed by induction up the tree. The previous theorem establishes the base case. Suppose there is a crossing in L which can be resolved to give two non-split, alternating branch loci, L_0, L_1. Suppose both of these satisfy the conclusion of the theorem.

One of L_0 or L_1 corresponds to two vertical strands. If we join these with an unknotted arc, the arc lifts to a circle in the branched double cover. The other resolution corresponds to removing a tubular neighborhood of this circle and re-gluing with a different framing. $\Sigma(L)$ is obtained by this framing plus/minus the meridian (depending upon the crossing).

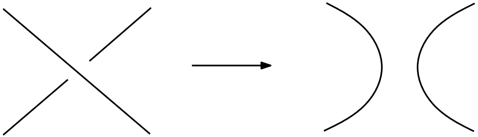
As an illustration



Seifert 0-framing corresponds to



Whereas Seifert -1-framing corresponds to



Let $\gamma \subset (Y, S)$ be a framed knot with $\gamma \pitchfork S = 0$. Then surgery on γ with this framing induces cobordism maps

$$\widehat{F} : \widehat{HFK}(Y, K, i) \longrightarrow \widehat{HFK}(Y_\gamma, K', i)$$

which fit into a long exact sequence (P. Ozsváth & Z. Szabó).

If we have three framings a, b, c where a is the meridian, and c is b plus the meridian, we obtain

$$\widehat{HFK}(Y_c, K_c, i) \cong \mathrm{MC}(\widehat{HFK}(Y_a, K_a, i) \xrightarrow{\widehat{F}} \widehat{HFK}(Y_b, K_b, i))$$

and likewise for cylic permutations of a, b, c. This strengthens P. Ozsváth & Z. Szabó result in a manner similar to how they strengthened the original long exact sequence.

Applying this to our setting we get the following inequality

$$|a_i(L)| \leq \operatorname{rank}(\widehat{HFK}(\Sigma(L), K, i) \leq \operatorname{rank}(\widehat{HFK}(\Sigma(L_0), K_0, i)$$

$$+\operatorname{rank}(\widehat{HFK}(\Sigma(L_1), K_1, i) = (-1)^i(a_i(L_0) + a_i(L_1))$$

where the respective inequalities come from

- The Euler characteristic property,
- The mapping cone property,
- The induction assumption

We need only show that $a_i(L) = a_i(L_0) + a_i(L_1)$ to be done.

We already know, from P. Ozsváth and Z. Szabó's work on double branched covers of alternating non-split links, that

$$|H_1(L;\mathbb{Z})| = |H_1(L_0,\mathbb{Z})| + |H_1(L_1,\mathbb{Z})|$$

when L is non-split, alternating. We will use this to determine the gradings.

Choose a surgery presentation of $(\Sigma(L_b), K_b)$. Then $(\Sigma(L_c), K_c)$ can be obtained by -1-surgery on a merdian of γ. $(\Sigma(L_a), K_a)$ can be obtained by another surgery: -1-surgery on a meridian of the previously added -1-framed curve.

Take the $\mathbb{Z}$-covering space corresponding to $PD[S]$. The surgery presentation lifts to one for this covering space. We look at the $\mathbb{Z}[T^{\pm 1}]$ intersection form for the handles. This has the form

$$M_a(T) = \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{M_b(T)} & & \\ & & 1 & 0 \\ \hline & 1 & -1 & 1 \\ & 0 & 1 & -1 \end{array}\right]$$

Then $\det(M_a(T)) = -\det(M_b(T)) - \det(M_c(T))$. If we plug in $T = 1$, we get $\pm|H_1(L_0, \mathbb{Z})| \pm |H_1(L_1, \mathbb{Z})| = |H_1(L; \mathbb{Z})|$. The signs depend upon the particular $\mathbb{Z}/2\mathbb{Z}$-gradings for the *spin*^{c} structures on the manifolds, but we happen to know what the signs must be, and can therefore determine the gradings. Appplying this to the $\mathbb{Z}[T^{\pm 1}]$-determinants yeilds the desired relationship. $\Diamond$

Reltionship to Khovanov homology

The resolution process involved in the previous proof has appeared

- in S. Wehrli's work on a spanning tree approach to Khovanov homology
- in P. Ozsváth and Z. Szabó's work connecting $\widehat{HF}(\Sigma(L))$ to Khovanov homology.

The connection is expressed by asking what you get if you resolve all crossings simultaneously, either as an *a* or *b*-resolution..

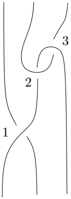
The short answer is that you get a spectral sequence. The (E^1, d^1)-page can be identified, adapting P. Ozsváth and Z. Szabó's work, with a version of Khovanov homology when everything is calculated using $\mathbb{F}_2$-coefficients.

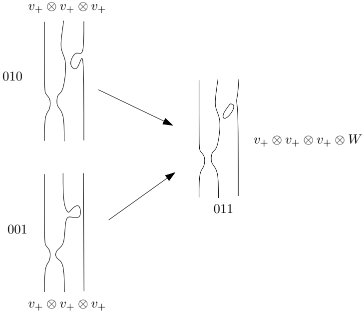
Namely, consider the full resolutions of L, sitting in the annulus that is the complement of the points where B meets the plane. M. Asaeda, J. Przytycki, and A. Sikora describe a khovanov homology for this situation, and the requisite (E^1, d^1)-page is the chain groups for this homology.

Their homology theory has an additional $\mathbb{Z}$-grading. In fact, B filters the original Khovanov complex, and the Asaeda-Przytycki-Sikora homology theory is the E^1-page of the spectral sequence corresponding to that filtration.

If this sounds confusing, it's because the complexes are actually bi-filtered, and we are looking at the two filtrations separately. However, in the case when L is alternating it turns out that the reduced Khovanov homology is identical to the knot Floer homology of K in the double branched cover, *and the filtration on the khovanov homology from B is identical to the filtration on* $\widehat{HF}(\Sigma(L))$ *induced by* K

(Well ... there's a caveat: the relationship is actually between K and the Khovanov homology of $-L$, the mirror of L.)





The two $v_+ \otimes v_+ \otimes v_+$ are mapped to $v_+ \otimes v_+ \otimes v_+ \otimes w_-$ leaving their sum, and $v_+ \otimes v_+ \otimes v_+ \otimes w_+$ as closed elements. This gives rank 2 homology groups, which corresponds to the knot Floer homology groups in index $+1$ for the knot 6_1, which is the double cover of B for this branch locus.

For L non-split, alternating there is also a result for the skein Khovanov homology, which generalizes that of E. S. Lee:

Theorem (L.R.)

Let $\mathbb{L}$ be an alternating link in $A \times I$ intersecting the spanning disc for B in an odd number of points. Then the Khovanov skein homology $H^{i;jk}(\mathbb{L})$, over $\mathbb{F}_2$, is trivial unless $k - j + 2i = \sigma(\mathbb{L})$, and the signature of the oriented link, $\sigma(\mathbb{L})$ is calculated thinking of L as embedded in S^3.

A consequence of this theorem and the theorem we discussed today is that for L non-split, alternating the spectral sequence collapses at E^2 for each i.

The knot determined by the example of branching over 4_1: