# Birman-Craggs-Johnson Homomorphism of the Torelli group

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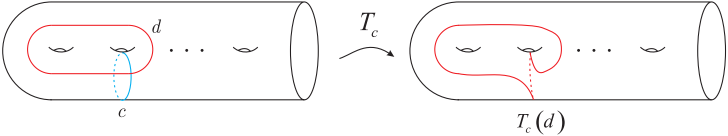
# Mapping Class Group Basics

**Definition**

The **Mapping Class Group** of a surface $S$ is the group of orientation preserving automorphisms of $S$ up to isotopy, denoted $Mod(S)$.

**Definition**

A **Dehn Twist** about a simple closed curve (scc), $c$, is denoted $T_c$.




**Fact:** $Mod(S)$ is finitely generated by Dehn Twists about scc's.

# The Torelli Subgroup

## Definition

The **Torelli subgroup**, $\mathcal{I}$, is the subgroup of $Mod(S)$ that acts trivially on $H_1(S, \mathbb{Z})$.

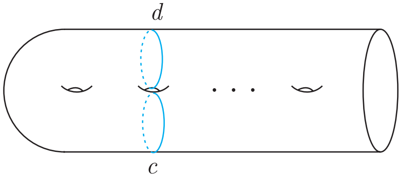



Figure: BP map $T_c T_d^{-1}$

## Definition

A **bounding pair map** (BP) is opposite twists about two disjoint, non seperating, homologous scc's in $S$.

# More on the Torelli Subgroup

Other elements in $\mathcal{I}$ are twists about separating curves.

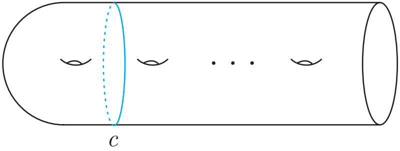



Figure: Separating curve

**Fact:** $\mathcal{I}$ is generated by Dehn twists about separating curves and BP maps.

**Definition**

The **Johnson Kernel**, $\mathcal{K}$, is the subgroup generated by twists about separating curves.

# Commutator Pairs

## Definition

A **commutator pair** is $[T_c, T_d]$ where $c$ and $d$ are simple closed curves with $\hat{i}(c, d) = 0$ and $i(c, d) = 2$

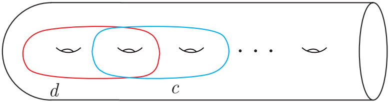



Figure: Commutator Pair $[T_c, T_d]$

Commutator pairs are in $\mathcal{I}$ because:

$$\begin{aligned} [T_c, T_d] &= (T_c)(T_d T_c^{-1} T_d^{-1}) \\ &= T_c T_{T_d(c)}^{-1} \end{aligned}$$

Curves $c$ and $T_d(c)$ are homologous

$\Rightarrow T_c T_{T_d(c)}^{-1} \in \mathcal{I}$

# Questions about $\mathcal{I}$

- **Known:** $\mathcal{I}$ is finitely generated by BP maps when $g \geq 3$. The order of the set is exponential in the genus.
- **Question:** What are other generating sets?
- **Question:** Can we find smaller ones?
- **Question:** What is the subgroup generated by commutator pairs? Is it $\mathcal{I}$?
- **Question:** Is $\mathcal{I}$ finitely presentable?

# The Rochlin Invariant

## Definition

Let $W$ be a homology sphere and $X$ be a simply connected parallelizable 4-manifold so that $W = \partial X$.

1. Such an $X$ always exists
2. signature($X$) is divisible by 8
3. $\frac{\mathrm{signature}(X)}{8} \bmod 2$ is independent of $X$.
4. $\Rightarrow$ it is an invariant of $W$, called the **Rochlin Invariant.**

# Birman-Craggs Homomorphisms

## Definition

(1978) The **Birman-Craggs homomorphisms** are a collection of homomorphisms

$$\rho_h : \mathcal{I} \to \mathbb{Z}_2$$

defined as follows:

1. Choose an embedding $h : S \hookrightarrow S^3$ and identify $S$ with $h(S)$.
2. For $f \in \mathcal{I}$, split $S^3$ along $S$ and reglue the two pieces using $f$ creating $W(h, f)$.
3. Since $f$ acts trivially on $H_1(S, \mathbb{Z})$
   $\Rightarrow$ 3-manifold $W(h, f)$ is a homology sphere.
   $\Rightarrow$ Rochlin invariant $\mu(h, f) \in \mathbb{Z}_2$ is defined.
4. Fix $h$ (the embedding) then

$$\rho_h(f) = \mu(h, f)$$

# Seifert Linking Form

It turns out that $\rho$ is not very sensitive to the embedding $h$.

- Any surface $S \subset S^3$ has an associated bilinear **Seifert Linking Form** $L(\alpha, \beta)$ for $\alpha, \beta \in H_1(S, \mathbb{Z})$.
- Then
  1. Take the bilinear Seifert Linking form $L(\alpha, \beta)$
  2. Use $\mathbb{Z}_2$ coefficients
  3. Set $\alpha = \beta$

  This gives a **mod 2 self-linking form** $\omega_h$ on $H_1(S, \mathbb{Z}_2)$
- Given an embedding $h : S \hookrightarrow S^3$ and identifying $S$ with $h(S)$ we induce a self-linking form $\omega_h$ on $H_1(S, \mathbb{Z}_2)$

# BC Homomorphism Facts

**Theorem**

*$\rho_1 = \rho_2 \iff h_1$ and $h_2$ induce the same mod 2 self-linking form on S.*

- $\Rightarrow \rho_h$ is only dependent on the quadratic form $\omega$ induced by $h$, so we could call it $\rho_\omega$.
- These forms are completely classified and yield the following facts.

Facts about BC-homomorphisms:

1. The number of BC-homomorphisms $\mathcal{I}_{g,1} \to \mathbb{Z}_2$ is $2^{2g}$.
2. $\{\rho_\omega\}$ span a $\mathbb{Z}_2$ vector space of dimension

$$\binom{2g}{3} + \binom{2g}{2} + \binom{2g}{1} + \binom{2g}{0}.$$

Two ways to think of BC-homomorphisms

1. Fixing the embedding $h$ we get $\rho_h : \mathcal{I} \to \mathbb{Z}_2$ where

$$\rho_h(f) := \mu(h, f) = \mu(\omega, f)$$

In a sense BC-homomorphisms are $\mu(\omega, -) : \mathcal{I} \to \mathbb{Z}_2$

2. Fixing $f \in \mathcal{I}$, consider $\sigma_f : \Psi \to \mathbb{Z}_2$ where

$$\sigma_f(\omega) := \mu(\omega, f)$$

So we have $\mu(-, f) : \Psi \to \mathbb{Z}_2$

# Combining BC-homomorphisms

In a sense $\sigma$ is the "dual" version of $\rho$.

$$\rho_\omega(f) = \mu(\omega, f) = \sigma_f(\omega)$$

Let $\mathcal{C} = \bigcap \text{ker}\rho_\omega$, then

$$\sigma_f = 0 \iff f \in \mathcal{C}$$

Then we can define $\sigma$ as follows:

$$\sigma : \mathcal{I} \to \{\text{vector space of functions } \Psi \to \mathbb{Z}_2\}$$

**Note:** ker $\sigma = \mathcal{C}$.

# Boolean Polynomials

Construct from $H_1(S, \mathbb{Z}_2)$ a $\mathbb{Z}_2$-algebra $B$ such that:

1. $B$ is commutative with unity
2. Generated by $\bar{a}$ where for nonzero $a \in H_1(S, \mathbb{Z}_2)$, get function

$$\bar{a} : \Omega \to \mathbb{Z}_2$$

$$\bar{a}(\omega) = \omega(a)$$

3. $\bar{a}^2 = \bar{a} \quad \forall\, a \neq 0$ in $H_1(S, \mathbb{Z}_2)$.
4. $\overline{(a+b)} = \bar{a} + \bar{b} + a \cdot b$ where $a \cdot b \in \mathbb{Z}_2 \subset B$.

This shows the connection with quadratic forms:

$$\omega : H_1(S, \mathbb{Z}_2) \to \mathbb{Z}_2$$

$$\omega(a+b) = \omega(a) + \omega(b) + a \cdot b$$

Let $B_k$ be the vector space of all elements of degree at most $k$.

# Birman-Craggs-Johnson Homomorphism

Johnson (1980) combined all BC homomorphisms into one surjective homomorphism

$$\sigma : \mathcal{I}_{g,1} \to B_3$$

Note:

1. For $h \in Mod(S_{g,1})$ and $f \in \mathcal{I}$

$$\sigma(hfh^{-1}) = h(\sigma(f))$$

2. BC homomorphisms generate $B_3^* = Hom(B_3, \mathbb{Z}_2)$

Computation of BCJ for genus $k$ separating curve:

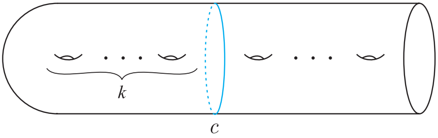



Figure: Genus $k$ separating curve

- Choose a symplectic basis $\{a_1, \cdots, a_k, b_1, \cdots, b_k\}$ for the subsurface bounded by $c$.
- $\sigma(T_c) = \sum_{i=1}^{k} \bar{a}_i \bar{b}_i$
- Johnson showed this is independent of choice of symplectic basis.

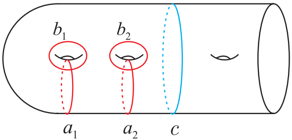
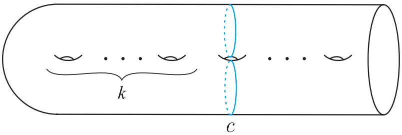



Figure: $\sigma(T_c) = \bar{a}_1 \bar{b}_1 + \bar{a}_2 \bar{b}_2$

Computation for BCJ of BP map $T_c T_d^{-1}$:




- Choose a symplectic basis $\{a_1, \cdots, a_k, b_1, \cdots, b_k\}$ for the subsurface bounded by $c$ and $d$.
- $\sigma(T_c T_d^{-1}) = \sum_{i=1}^{k} \bar{a}_i \bar{b}_i (1 - \bar{c})$
- Again this is independent of choice of basis.

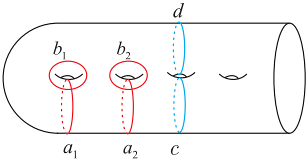



Figure: $\sigma(T_c T_d^{-1}) = (\bar{a}_1 \bar{b}_1 + \bar{a}_2 \bar{b}_2)(1 - \bar{c})$

# Separating Commutator Pairs

## Definition

A **separating commutator pair** is a commutator pair where at least one curve is separating.

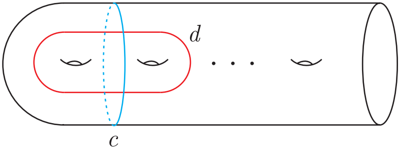



Figure: Commutator Pair with $c$ a separating curve

Separating commutator pairs are in $\mathcal{K}$ because $c$ and $T_d(c)$ are both separating curves.

Do separating commutator pairs generate $\mathcal{K}$?

- BCJ will help us.
- **Known:** $\mathcal{K}$ maps onto $B_2$.
- **Question:** What is the image of separating commutator pairs under BCJ?

**Theorem (Childers)**

*The image of the subgroup generated by separating commutator pairs under $\sigma$ is* $\langle 1, \overline{a_i}, \overline{b_i}, \overline{a_i}\overline{b_j}, \overline{a_i}\overline{b_i} + \overline{a_j}\overline{b_j} | 1 \leqslant i, j \leqslant g, i \neq j \rangle$

Sketch of proof:

$$\begin{aligned}\sigma(T_c T_d T_c^{-1} T_d^{-1}) &= \sigma(T_c) + \sigma(T_d T_c^{-1} T_d^{-1}) \\ &= \sigma(T_c) + T_d \cdot \sigma(T_c^{-1}) \\ &= \underbrace{\sigma(T_c)}_{\text{has symplectic term } \bar{a}_i \bar{b}_i} + \underbrace{T_d \cdot \sigma(T_c)}_{\text{has symplectic term } (\overline{T_d(a_i)})(\overline{T_d(b_i)})}\end{aligned}$$

So the symplectic terms come in pairs.

# Special Commutator Pairs

- Consider a regular neighborhood of a commutator pair.
- It is a lantern (ie- $S_{0,4}$).
- If one of the boundary components is separating, then it is a **special commutator pair**.

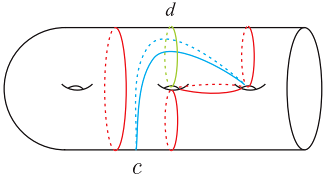



Figure: Special Commutator Pair $[T_c, T_d]$

- Then $[T_c, T_d]$ can be rewritten as $[A, C]$ where $A, C \in \mathcal{I}$.
- So $\sigma([T_c, T_d]) = 0$
- **Question:** What commutator pairs are in the kernel of BCJ?

# References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