Embedded Khovanov Homology

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Joint LSU-Iowa Topology Seminar, 2007

Outline

Webs

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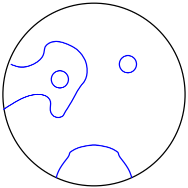
- Let $F \subset \partial M$ be a compact surface and let $B \subset F$ be a finite set of points. Let $G = \overline{\partial M - F}$.
- A web in F with boundary B is a properly embedded compact 1-manifold in F with boundary B



Pre-Foams

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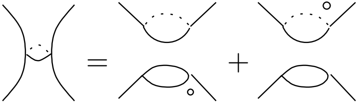
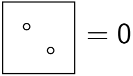
- Let w and v be webs in F and G with the same boundary B.
- A prefoam is an isotopy class of marked surfaces in M with boundary $v \cup w$
- The markings consist of finite collections of points in S.
- The degree of prefoam S is $2d - \chi(S) + \frac{1}{2}|B|$, where d is the total number of points in the marking of S.
- Let $P(v, w)$ be the free $\mathbb{Z}$-module on the prefoams with boundary $v \cup w$.

Bar-Natan Relations

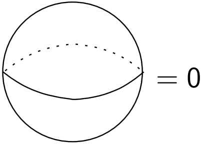
- The neck cutting relation.



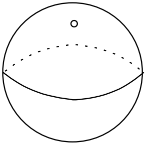
- Two dots on the same component are zero.



- A sphere bounding a ball without a dot is zero.



- A sphere with a dot bounding a ball is 1.



$= 1$

The Bar-Natan module

- Let $R(v, w)$ be the submodule of $P(v, w)$ spanned by relations.
- We define $C(v, w) = P(v, w)/R(v, w)$.
- As the relations preserve degree, $C(v, w) = \oplus_d C_d(, w)$, where $C_d(v, w)$ is the part of the module of degree d.
- An element of $C(v, w)$ is called a *foam*. If $v = \emptyset$ we just call it $C(w)$, if $w = \emptyset$ too, we call it $C(M)$.

Exploring the neck cutting relation

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- If M is irreducible (every sphere bounds a ball), then the set of incompressible surfaces without dots is linearly independent in $C(v, w)$.
- Relations between surfaces with dots seem to come from product regions between surfaces.

An Example

- Consider two disjoint incompressible tori, $S = S^1 \times S^1 \times \{1\}$ and $S' = S^1 \times S^1 \times \{-1\}$ in $S^1 \times S^1 \times S^1$.

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- Let N be the surface which is the result of tubing S and S' by a single uknotted tube.
- The neck cutting relation says $N = F + F'$.

On the other hand N is a surface of genus two bounding a handlebody. Applying the neck cutting relation along two disjoint nonseparating curves on N yields a sphere with two dots, which is zero.
So, $F = -F'$.

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- The Bar-Natan skein module of $S^2 \times S^1$, $C(S^2 \times S^1)$. has basis consisting of e_n, the disjoint union of n incompressible spheres without a dot if $n > 0$ and the empty set if $n = 0$ and a foam which is a single incompressible sphere with a single dot.

- If the ranks of the $C_d(v, w)$ are finite we can form the Bar-Natan polynomial.

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- The Bar-Natan polynomial of $S^2 \times S^1$ is not a polynomial, it is a rational function,

$$p(S^2 \times S^1) = 1 + \frac{1}{1-q^2}.$$

Conjecture

If M is closed, irreducible and atoroidal, then $p(M)$ is a rational function.

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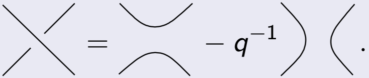
- It is easy to see $C(w)$ has the Seifert surface and the Seifert surface with a dot as basis. Hence the Bar-Natan polynomial is $q + q^3$.
- Notice that the Bar-Natan polynomial of a knot captures the genus of the knot. If the knot isn't fibered there will be more terms, in fact it will capture all the incompressible Seifert surfaces of the knot.

The Skein relation

We want to extend the Bar-Natan polynomial to link diagrams in the boundary of a three-manifold so that it satisfies the following skein relation.

Kauffman brack ala Khovanov



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- Finally,

$$p(D) = \sum_{s} (-q)^{I(s)} p(s).$$

The polynomial is invariant under the second and third Reidemeister moves and transforms as you would expect under the first Reidemeister move. This all depends on the following proposition.

If $M = M_1 \cup M_2$ and $M_1 \cap M_2$ is a disk in the boundary of each, and $w = w_1 \cup w_2$ is a web in ∂M so that $w_1 = w \cap M_1$ and $w_2 = w \cap M_2$ then,

$$C(M, w) = C(M_1, w_1) \otimes C(M_2, w_2).$$

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- The Bar-Natan polynomial is in general a power series. Its terms are in one to one correspondence with the terms in the sum to compute the Yang-Mills measure.

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- We form the standard cube of modules corresponding to states, where

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- At the ith crossing, if s and s' are states that differ only at the ith crossing, where s has the 1 smoothing and s' has the 0 smoothing, then there is a map $d_i : C(s, v) \to C(s', v)$ coming from gluing a saddle into each surface with boundary $s \cup v$ so that the new surface has boundary $s' \cup v$. The shift in polynomial gradings makes this degree preserving. The boundary operator is the signed sum of the d_i's.

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- This theory catagorifies $p(D, v)$.