

MATHEMATICS AND ITS HISTORY

Jimmie Lawson

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Chapter 1

Mathematics of Ancient Egypt

1.1 History

Egyptian mathematics dates back at least almost 4000 years ago. The main sources about mathematics in ancient Egypt are the “Moscow Papyrus,” dating back to around 1850 B.C., and “Rhind’s Papyrus” in the British Museum, which was copied from the original in 1650 B.C. by someone named Ahmose. Egyptian mathematics apparently arose from various practical concerns such as surveying of fields after Nile flooding, construction projects such as the pyramids, the making of calendars, and various accounting activities and was practiced by scribes and perhaps priests.

The two papyri contain various specific elementary mathematical problems (for example: how many bricks does it take to build a ramp of certain dimensions?) and their solutions (often worked out). Although the problems often go well beyond practical concerns, we find no general methods of solutions described and nothing that could be considered theoretical.

The Egyptians used a base 10 grouping system (with symbols representing various powers of 10 grouped together), but the scribes also developed a ciphered system (with symbols for all numbers of the form $k \cdot 10^n$ for $1 \leq k \leq 9$), which made writing of numbers much less cumbersome. They used fractions which they curiously insisted on representing as sums of integer reciprocals, i.e., fractions with numerator 1 (for example, Ahmose writes $\frac{2}{45} = \frac{1}{30} + \frac{1}{90}$). The methods used to rewrite fractions in this form are not clearly understood.

Adding and subtracting are easy in grouping systems. The Egyptians had

a multiplication algorithm that reduced multiplication to repeated doubling of one multiplicand and a final addition. They used a modified version of this algorithm for division.

The Egyptian papyri present various word problems that involve solving linear equations and various methods for solution, including one similar to that we use today. Another technique was that of *false position*, where a convenient but incorrect answer is assumed and then appropriated adjusted to obtain the correct one. The solutions are much more challenging since no modern algebraic symbolism is used, only descriptions in words of what is being done, and since the Egyptian treatment of fractions was so cumbersome.

In geometry the papyri demonstrate a knowledge of basic area and volume formulas. One interesting formula is that used for the area of a circle, given the diameter: $A = [(8/9)d]^2$. The most impressive formula obtained is that for the volume of a truncated pyramid with square base of side a and square top of side b : $V = \frac{1}{3}h(a^2 + ab + b^2)$. It remains unknown how the Egyptians found this formula.

1.2 Problems

1. Rewrite the following numbers in hieroglyphics and add:
(i) $234 + 765$ (ii) $4,555 + 5,648$ (iii) $36,486 + 9,018$
2. The “doubling” method of Egyptian multiplication requires writing any whole number as a sum of powers of two. How do you know that this can be done for any number? Use the reverse remainder algorithm to write the following numbers as sums of powers of two:
(i) 73 (ii) 52 (iii) 98 (iv) 151
3. Multiply using the Egyptian method of doubling:
(i) 19×29 (ii) 25×73 (iii) 71×211
4. The validity of the Egyptian multiplication algorithm depends on the ability to write a number as a sum of powers of 2 and what basic arithmetic law?
5. The Egyptian multiplication algorithm can also be used with negative powers of 2. Use this idea to show the product of $1 + \frac{1}{2} + \frac{1}{4}$ by $\frac{1}{14}$ is equal to $\frac{1}{8}$ (Problem 12 of the Rhind Papyrus).
6. Perform the following divisions using the Egyptian method:
(i) $96 \div 8$ (ii) $805 \div 35$ (iii) $84 \div 5$
7. The Egyptians wrote their fractions as sums of *distinct* unit fractions. Historians are unsure about how the Egyptians reduced fractions to this form. One method for doing this that has been referred to as the *splitting method* depends on the identity

$$\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)}.$$

For example,

$$\frac{2}{19} = \frac{1}{19} + \frac{1}{19} = \frac{1}{19} + \frac{1}{20} + \frac{1}{380}.$$

For fractions with larger numerators, the splitting may need to be repeated.

- (i) Verify the algebraic identity behind the splitting method.
- (ii) Represent $\frac{3}{7}$ and $\frac{4}{15}$ as sums of distinct unit fractions by using (a) the splitting method and (b) Fibonacci’s method.
8. The *Method of False Position* is a guess method for solving *linear* equations. For example, a number plus its double plus its third equals 20. What is the number? We make some initial guess, typically one that is easy to evaluate. If we guess 12, then we obtain $12 + 2(12) + (12/3) = 12 + 24 + 4 = 40$. We

then from the fraction (desired answer)/(answer from guess)= 20/40 = 1/2 and multiply it by our first guess $(1/2) \times 12 = 6$ to get the desired answer.

(i) Use the method of false position to solve

$$3x - \frac{x}{4} + \frac{3}{8}x = 10$$

(ii) Problem 32 of the Rhind Papyrus states that a quantity, its third, and its fourth added together yield 2. Use the method of false position to find the quantity. Express your answer in Egyptian fashion.

9. The Moscow and Rhind Papyri give what are probably the earliest remaining records of area and volume formulas. Problem 41 of the Rhind Papyrus asks for the volume of grain that can be stored in a cylindrical granary of diameter 9 cubits and height 10 cubits. Solve the problem using the Egyptian value $\pi = 4(8/9)^2$.

10. Problem 58 of the Rhind Papyrus asks for the “seked” of a pyramid (with square base) if it is $93\frac{1}{3}$ cubits high and the side of its base is 140 cubits. (Explanation: The seked of an isosceles triangle is given by $s \div 2h$ where s is the length of the base and h is the height or altitude on the base. For the pyramid in question the seked would be the seked of a isosceles slice of the pyramid with base connecting the midpoints of opposing sides.)

(i) Verify that the seked of an isosceles triangle equals the cotangent of one of the base angles.

(ii) Solve Problem 58. How does this compare to the slope of a lateral face of the pyramid? (The seked in this problem has been associated with the slope of the lateral faces of the Second Pyramid at Gizeh.)

11. The Moscow Papyrus (1850 B.C.) shows that the Egyptians were familiar with the correct formula for the volume of a truncated pyramid:

$$V = \frac{h}{3}(a^2 + ab + b^2),$$

where h is the altitude, a is the length of the square base, and b is the length of the square top.

(i) Solve Problem 14, the volume of a truncated pyramid of vertical height 6, a side of length 4 for the base, and a side of length 2 on the top.

(ii) Show that the Egyptian formula is equivalent to one known to the Babylonians:

$$V = h \left[\left(\frac{a+b}{2} \right)^2 + \frac{1}{3} \left(\frac{a-b}{2} \right)^2 \right].$$

Chapter 2

Euclid's *Elements*

2.1 Problems

A very useful web reference for Euclid's *Elements* is the web version prepared by D. E. Joyce at the Clark University site.

1. Euclid's notion of "equal" is equality of some geometric measurement (such as length or area). It corresponds in modern math to what we now call an *equivalence relation*. (What we would call "equality" Euclid calls "coinciding.")

Suppose that we use \cong for the relation "equal" in Euclid's Common Notions. Define what it means for \cong to be an equivalence relation and show that Common Notions 1 and 4 may be interpreted to show that \cong is an equivalence relation. In his 1899 reformulation of Euclid's geometry, Hilbert defined equality of length by postulating an equivalence relation on line segments.

2. Give Pappus' proof that the base angles of an isosceles triangle are equal (Proposition I.5) (Hint: show that flipping the triangle gives a congruence of the triangle with itself.)
3. The assertion that if two lines cut one another, then they make vertical angles that are equal (Proposition I.15) has been attributed to Thales. Prove it by appealing to Proposition I.13, which says that if a ray is drawn from a point on a line, then the supplementary angles that are formed equal two right triangles (180°).

4. Show that the angles of a triangle sum to two right angles (180°) by constructing a line through a vertex parallel to the opposite side.
5. Prove that if the opposite sides of a quadrilateral are equal and parallel, then the quadrilateral is a parallelogram (Proposition I.33). (Hint: Draw a diagonal and prove the triangles are congruent.)
6. Illustrate with a square of side having length $a + b$ that $(a + b)^2 = a^2 + 2ab + b^2$ (see Proposition II-4).
7. Describe how to carry out the following constructions:
 - (i) the perpendicular bisector of a line segment;
 - (ii) the tangent to a circle.

How does Euclid define a tangent line to a circle? How does that differ from the way we define it in calculus? Show that your construction in part (ii) satisfies Euclid's definition.

8. Given a line segment of length a , describe the construction for obtaining x such that $a : x = x : (a - x)$ (Proposition II-11). Show that $a : x = \sqrt{5} + 1 : 2$, the **golden ratio**.
9. Show that Proposition II-13 is equivalent to the law of cosines for an acute-angled triangle: In acute-angled triangles the square on the side opposite the acute angle is less than the sum of the squares on the sides containing the acute angle by twice the rectangle contained by one of the sides about the acute angle, namely that on which the perpendicular falls, and the straight line cut off within by the perpendicular towards the acute angle.
10. Describe the construction for inscribing a regular hexagon in a circle. Explain why it works.
11. Explain using the sum of angles at a vertex why there can be at most 5 regular polyhedra.

Chapter 3

Test 1 Review

3.1 Study Guide

Material Covered: Chapters I-III of text and handouts

1. Proofs to know: (i) $\sqrt{2}$ is irrational. (ii) there is no largest prime.
2. Definitions and facts to know: triangular, square and oblong numbers, commensurable and incommensurable lengths, ruler and compass constructions, similar triangles, congruent triangles, ways of defining the conic sections, prime number, proportion, definition of a regular polyhedron + description of 5 possibilities, (Eudoxos') method of exhaustion, spiral of Archimedes, Apollonius tangent circle problem and special cases thereof, three famous construction problems of Greek mathematics, explanation of the axiomatic, deductive, idealized, and constructive nature of Greek math.
3. Algorithms: Egyptian multiplication, writing numbers in base 60 (for example, $(10, 30; 20)_{60} = 630\frac{1}{3}$), finding Pythagorean triples, Euclidean algorithm for gcd, back substitution for writing the gcd d of m, n in the form $d = am + bn$, applications of the law of the lever, summing finite and infinite geometric series, angle size in regular polygons, computation of basic areas and volumes we have studied.
4. Famous theorems to state and understand: Pythagorean theorem and converse, Fundamental Theorem of Arithmetic, characterization of regular polyhedra, three methods of showing triangles are congruent (SAS,

SSS, ASA) basic theorem for showing triangles are similar (two angles are congruent), tangent to circle perpendicular to diameter.

5. For Euclid, you should know the material, proofs, and problems, in the previous problem handout on Euclid's *Elements*.
6. Discussion topics: Two of the following four topics will be given, and you may choose one to write on: overview of Egyptian mathematics, overview of Babylonian mathematics, overview of Pythagorean mathematics, overview of the mathematics of Archimedes.