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MULTIPLES OF WEIERSTRASS POINTS AS SPECIAL DIVISORS

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ABSTRACT. Complex spacss $\mathcal{W}_n^r$ of Weierstrass points are isomorphic to the intersection, on the nth symmetric product of the universal curve over the Teichmüller space, of complex spaces $\mathcal{G}_n^r$ of special divisors with the diagonal Δ_n consisting of divisors which are multiples of a point. The tangent space at a point of this intersection is described and it is shown that $\mathcal{G}_n^1 - \mathcal{G}_n^2$ and Δ_n intersect transversally.

Let $T = T_g$ denote the Teichmüller space for Teichmüller surfaces of genus $g > 1$ and let $\pi: V \to T$ denote the universal curve of genus g. Denote by $V_T^{(n)}$ the nth symmetric product of V over T. Let $\mathcal{G}_n^r$ denote the closed complex subspace of $V_T^{(n)}$ whose points are divisors of degree n and projective dimension at least r (see [3], [2]). We have proved

THEOREM 1 ([3]). *Suppose $n \leqslant g$. Then $\mathcal{G}_n^1 - \mathcal{G}_n^2$ is smooth of pure dimension $2n + 2g - 4$.*

For $2 \leqslant n \leqslant g$, let $\mathcal{W}_n^r$ denote the closed complex subspace of V consisting of those $(t, P) \in V$ such that there are at least r gaps less than or equal to n in the Weierstrass gap sequence at P on V_t. These spaces were introduced in [4] and, by employing methods similar to those used in the proof of Theorem 1, we proved

THEOREM 2 ([4]). *For $2 \leqslant n \leqslant g$, $\mathcal{W}_n^1 - \mathcal{W}_n^2$ is smooth of pure dimension $n + 2g - 3$.*

In this note, we describe the relationship between the $\mathcal{G}_n^r$ and the $\mathcal{W}_n^r$ and show how Theorem 2 may be derived in a direct fashion from Theorem 1.

Let Δ_n denote the image of $\delta_n: V \to V_T^{(n)}$, the closed immersion which takes a point (t, P) to the point (t, nP). The following proposition follows easily from the definitions.

PROPOSITION 1. *For $2 \leqslant n \leqslant g$,*

$$\delta_n|_{\mathcal{W}_n^r}: \mathcal{W}_n^r \xrightarrow{\cong} \mathcal{G}_n^r \cap \Delta_n.$$

We now explicitly consider the intersection of $\mathcal{G}_n^r$ and Δ_n. Suppose $(t, nP) \in \mathcal{G}_n^r \cap \Delta_n$. Put $X = V_t$. Let z be a local coordinate on X centered at P and let $z_1, \ldots, z_n$ denote n copies of z. Let $\sigma_1, \ldots, \sigma_n$ denote the n elementary symmetric functions in $z_1, \ldots, z_n$. Let $c_1, \ldots, c_{3g-3}$ denote Patt's local coordinates on T

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centered at t (see [8]). Then $c_1, \ldots, c_{3g-3}, \sigma_1, \ldots, \sigma_n$ are local coordinates on $V_T^{(n)}$ centered at (t, nP). Put

Z = tangent space to $V_T^{(n)}$ at (t, nP);

Z_1 = tangent space to $\mathcal{G}_n^r$ at (t, nP);

Z_2 = tangent space to Δ_n at (t, nP).

We describe coordinates for Z. Suppose $\xi \in Z$. We may view ξ as a $\mathbf{C}$-homomorphism of local rings

$$\xi\colon \mathcal{O}_{V_T^{(n)},(t,nP)} \to \mathbf{C}[\varepsilon]/(\varepsilon^2)$$

(cf. [6, p. 332]). Then ξ is determined by its values on a set of local parameters of $\mathcal{O}_{V_T^{(n)},(t,nP)}$. So, if $\xi(c_m) = b_m\varepsilon$, $m = 1, \ldots, 3g-3$, and $\xi(\sigma_i) = u_i\varepsilon$, $i = 1, \ldots, n$, then $(u_1, \ldots, u_n, b_1, \ldots, b_{3g-3})$ serve as coordinates for Z.

Proposition 2. *$\xi = (u_1, \ldots, u_n, b_1, \ldots, b_{3g-3})$ is in Z_2 if and only if $u_2 = u_3 = \cdots = u_n = 0$.*

Proof. Suppose $(t_1, Q_1 + \ldots + Q_n) \in V_T^{(n)}$ is a point near (t, nP). Then $(t_1, Q_1 + \cdots + Q_n) \in \Delta_n \Leftrightarrow z(Q_1) = \cdots = z(Q_n) = z_0 \Leftrightarrow z_0$ is an n-fold root of

$$\begin{aligned} F(Y) = \prod_{i=1}^{n} (Y - z(Q_i)) &= Y^n - \sigma_1(z(Q_1), \ldots, z(Q_n))Y^{n-1} \\ &\quad + \cdots + (-1)^n\sigma_n(z(Q_1), \ldots, z(Q_n)) \\ &\Leftrightarrow F(z_0) = F'(z_0) = \cdots = F^{(n-1)}(z_0) = 0 \\ &\Leftrightarrow \sigma_k(z(Q_1), \ldots, z(Q_n)) = \binom{n}{k}[\sigma_1(z(Q_1), \ldots, z(Q_n))]^k/n^k \\ &\qquad\qquad \text{for } k = 2, 3, \ldots, n, \end{aligned}$$

and $\sigma_1(z(Q_1), \ldots, z(Q_n)) = nz_0$.

So, near (t, nP), Δ_n is defined by the equations $\{\sigma_k = \binom{n}{k}\sigma_1^k/n^k\}$, $k = 2, \ldots, n$. Thus ξ is tangent to Δ_n at (t, nP) if and only if $\xi(\sigma_k) = 0$ for $k = 2, 3, \ldots, n$.

We next recall the description of Z_1, which was given in [3]. Let $1, \gamma_2, \ldots, \gamma_g$ denote the Weierstrass gaps at $P \in X$. Choose a basis of holomorphic 1-forms $d\zeta_1, \ldots, d\zeta_g$ on X such that $\operatorname{ord}_P d\zeta_j = \gamma_j - 1$. Write

$$d\zeta_j = \sum_{i=0}^{\infty} a_{i,j}z^i\, dz.$$

For details concerning the following result, we refer the reader to [3].

Proposition 3. *Suppose $n \leqslant g$ and $\xi \in Z$. Then $\xi \in Z_1$ if and only if all minors of order $n - r + 1$ of the matrix*

$$\mathfrak{M} = \left[\begin{array}{c:c} (-1)^i a_{i,j} & \varepsilon\left[\displaystyle\sum_{l=1}^{n} (-1)^{i+l-1} a_{i+l}u_l + \sum_{m=1}^{3g-3} \tau'_{P,i}(Q_m)\zeta'_j(Q_m)b_m\right] \\ i = 0, \ldots, n-1 & i = 0, \ldots, n-1 \\ j = 1, \ldots, n-r & j = n-r+1, \ldots, g \end{array}\right]$$

vanish, where $\tau_{P,k}$ is an elementary integral of the second kind on X with pole of order $k+1$ at P and where $(Q_1, \ldots, Q_{3g-3})$ is any point chosen from an open subset of X^{3g-3}.

Now, suppose $(t, nP) \in \mathcal{G}_n^r - \mathcal{G}_n^{r+1}$, $n \leq g$. Then $\mathfrak{M}$ will have a nonzero minor of order $n - r$, call it μ, and in order that all minors of order $n - r + 1$ of $\mathfrak{M}$ vanish, it is sufficent that those minors of order $n - r + 1$ which contain μ should vanish. This gives rise to $r(g - n + r)$ linear equations $\{E_k\}$ in $u_1, \ldots, u_n$, $b_1, \ldots, b_{3g-3}$. These equations are of the form

$$E_k\colon \sum_{l=1}^{n} e_{k,l}u_l + \sum_{m=1}^{3g-3} \alpha_k(Q_m)b_m = 0,$$

where the α_k are (not necessarily finite) quadratic differentials on X. (The α_k arise from the products $d\tau_{P,i}d\zeta_j$ which appear in $\mathfrak{M}$–see [3].)

THEOREM 3. *Suppose* $n \leq g$, $r(g - n + r) \leq 3g - 3$, *and* $(t, nP) \in \mathcal{G}_n^r - \mathcal{G}_n^{r+1}$. *If the above* α_k, $k = 1, \ldots, r(g - n + r)$, *are linearly independent quadratic differentials, then*:

1) $\dim Z_1 = 3g - 3 + (r + 1)(n - r) - rg + r$ *and* $\mathcal{G}_n^r$ *is smooth at* (t, nP).

2) $\dim Z_1 \cap Z_2 = 3g - 2 - r(g - n + r)$ *and* $\mathcal{G}_n^r$ *and* Δ_n *intersect transversally at* (t, nP).

PROOF. One may show, as in [3], that if the α_k are linearly independent, then since $(Q_1, \ldots, Q_{3g-3})$ is any point from an open subset of X^{3g-3}, the matrix $[\alpha_k(Q_m)]$, $k = 1, \ldots, r(g - n + r)$ and $m = 1, \ldots, 3g - 3$, will have maximum rank. It then follows that the systems of equations which define Z_1 and $Z_1 \cap Z_2$ will have maximum rank, establishing the theorem.

We showed in [3] that at least $g - n + r$ of the α_k are linearly independent. In particular, if $r = 1$, then all the α_k are linearly independent. As a consequence we have

THEOREM 4. *Suppose* $(t, nP) \in \mathcal{G}_n^r - \mathcal{G}_n^{r+1}$, $n \leq g$. *Then*

(1) $\dim Z_1 \leq 2g + 2n - r - 3$; *in particular, if* $r = 1$, *then* $\dim Z_1 = 2g + 2n - 4$ *and* $\mathcal{G}_n^1$ *is smooth at* (t, nP).

(2) $\dim Z_1 \cap Z_2 \leq 2g + n - r - 2$; *in particular, if* $r = 1$, *then* $\dim Z_1 \cap Z_2 = 2g + n - 3$ *and* $\mathcal{G}_n^1$ *and* Δ_n *intersect transversally at* (t, nP).

COROLLARY. *For* $n \leq g$,

(1) $\dim \mathcal{W}_n^r \leq 2g + n - r - 2$;

(2) $\mathcal{W}_n^1 - \mathcal{W}_n^2$ *is smooth of pure dimension* $2g + n - 3$.

REMARKS. (1) The smoothness of $\mathcal{G}_n^1 - \mathcal{G}_n^2$ has also recently been demonstrated by Arbarello-Cornalba [1] and Namba [7].

(2) Arbarello-Cornalba [1] have shown that $\mathcal{G}_n^2 - \mathcal{G}_n^3$ is smooth, but it does not necessarily follow that the α_k are then linearly independent or that this space intersects Δ_n transversally.

(3) In [5], we defined $\mathcal{W}_n^r$ for $n > g$. The points of this space are those $(t, P) \in V$ such that there are at least r gaps greater than n in the gap sequence at $P \in V_t$. We showed that for $n > g$, $\mathcal{W}_n^1 - \mathcal{W}_n^2$ is smooth of pure dimension $4g - n - 3$. This result can also be obtained as above by considering the intersection of $\mathcal{G}_n^r$ and Δ_n for $n > g$, but we note that, by our definition of $\mathcal{W}_n^r$, for $n > g$, $\mathcal{W}_n^r = \delta_n^{-1}(\mathcal{G}_n^{n-g+r})$.

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