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# On Higher Order Weierstrass Points of the Universal Curve*

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## 1. Introduction

Let $X$ denote a compact Riemann surface of genus $g$. Let $K$ denote the canonical line bundle on $X$ and denote by $K^n$ the $n$-fold tensor power of $K$. Put $d_n = \dim_{\mathbb{C}} H^0(X, K^n) = (2n-1)(g-1)+\delta_{1n}$. At each point $P \in X$, there is a sequence of $d_n$ integers $1=\gamma_1(P)<\gamma_2(P)<\cdots<\gamma_{d_n}(P) \leqq 2n(g-1)+1$, called the sequence of $n$-gaps at $P$. An integer $\gamma$ is an $n$-gap at $P$ if and only if there exists $\theta \in H^0(X, K^n)$ with a zero of order $\gamma-1$ at $P$. The point $P$ is called an *n-fold Weierstrass point* if $\gamma_{d_n}(P) > d_n$ (cf. [14, 21]). The *multiplicity*, or weight, of an $n$-fold Weierstrass point $P$ is $\sum_{i=1}^{d_n} (\gamma_i(P)-i)$. If we count these points with multiplicity, then every compact Riemann surface of genus $g \geqq 2$ has $g^3-g$ 1-fold Weierstrass points and $g \cdot d_n^2$ $n$-fold Weierstrass points for $n>1$ (cf. [17, 18, 13]).

In §2, we proceed as in [12] and define complex spaces of $n$-fold Weierstrass points of the universal curve over the Teichmüller space. We then give lower bounds on the dimension of these spaces and on the moduli of Teichmüller surfaces with a given type of $n$-fold Weierstrass point. In §3, we give an example to show that the dimension of these spaces can exceed these lower bounds.

Our complex spaces are not necessarily reduced (for definitions, see [6]). If $Y$ is a complex space, then $|Y|$ will denote the underlying set of points.

## 2. Complex Spaces of *n*-fold Weierstrass Points

The following theorem is due to Grothendieck [8; Exposé 7, Theorem 3.1] (also see [10]):

**Theorem 1.** *There exists a complex space $T_g$ and a family $V$ of Teichmüller surfaces of genus g over $T_g$ which is universal in the following sense: for every family $W$ of Teichmüller surfaces of genus g over a complex space $S$, there exists a*

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*unique map $\Phi: S \to T_g$ such that $W$ is isomorphic (as a family of Teichmüller surfaces) to the pullback via $\Phi$ of $V$.*

$T_g$ is called the *Teichmüller space* (for Teichmüller surfaces of genus $g$) and $V$ is called the *universal curve of genus* $g$. The Teichmüller space is a smooth, irreducible, and simply connected complex space [8]. Let $\pi: V \to T_g$ denote the structural morphism.

Let $\Omega^1_{V/T_g}$ denote the holomorphic line bundle on $V$ of relative holomorphic 1-forms (cf. [8, Exposé 14]). For $n \geqq 1$, put $\mathscr{K}^n$ equal to $(\Omega^1_{V/T_g})^{\otimes n}$. Now, if $s \in T_g$, then

$$\pi_* \mathscr{K}^n \otimes \kappa(s) \cong H^0(V_s, K_s^n),$$

where $K_s$ denotes the canonical bundle on $V_s$, the fiber of $\pi$ over $s$. Hence, by [7], $\pi_* \mathscr{K}^n$ is a holomorphic vector bundle of rank $d_n$ on $T_g$. Then $\pi^* \pi_* \mathscr{K}^n$ is a holomorphic vector bundle on $V$ whose fiber over a point $(s, P) \in V$ is the vector space $H^0(V_s, K_s^n)$.

Let $J^k_{V/T_g}(\mathscr{K}^n)$ denote the holomorphic vector bundle on $V$ of rank $k+1$ of $k$-jets along the fiber of $\mathscr{K}^n$ (cf. [12]). We recall that to construct this bundle, we identify two germs $f_1 dt^n$ and $f_2 dt^n$ of sections of $\mathscr{K}^n$ at $(s, P)$ if $\partial^l f_1/\partial t^l(s, P) = \partial^l f_2/\partial t^l(s, P)$ for $0 \leqq l \leqq k$ (where $t$ denotes a local coordinate on $V_s$ at $P$). For $n \geqq 1$ and $1 \leqq k \leqq d_n - 1$, let

$$u_k^n: \pi^* \pi_* \mathscr{K}^n \to J^k_{V/T_g}(\mathscr{K}^n)$$

be the bundle map whose fiber over $(s, P)$ is the map taking a section of $H^0(V_s, K_s^n)$ to its $k$-jet at $P$.

Let $s_0 \in T_g$. Then there exists on open neighbourhood $U$ of $s_0$ and sections $\theta_1, \ldots, \theta_{d_n}$ of $\pi_* \mathscr{K}^n$ such that $\theta_1(s), \ldots, \theta_{d_n}(s)$ is a basis of $H^0(V_s, K_s^n)$ for all $s \in U$ (cf. [2]). Let $(s_0, P) \in V$ and let $t$ denote a local coordinate on $V_{s_0}$ at $P$. Writing

$$\theta_j(s) = f_j(s, t)\, dt^n,$$

we have that the map $u_k^n$ is given in a neighborhood of $(s_0, P)$ by the matrix

$$[\partial^i f_j/\partial t^i] \qquad i = 0, \ldots, k; \quad j = 1, \ldots, d_n.$$

Let $Z^r(u_k)$ denote the closed complex subspace of $V$ over which the map $u_k^n$ has rank $\leqq (k+1) - r$. Such spaces of singularities of bundle maps, together with their infinitesimal structure, were studied in [11]. Put

$$\mathscr{W}_k^r(\mathscr{K}^n) = Z^r(u_{k-1}^n).$$

*Remarks.* 1) We have chosen our subscripts here to agree with those in [12, 3], which means that they differ by one from the subscripts used in [18, 13].

2) $\mathscr{W}_k^{r+1}(\mathscr{K}^n) \subseteq \mathscr{W}_k^r(\mathscr{K}^n)$

3) $\mathscr{W}_{k-1}^r(\mathscr{K}^n) \subseteq \mathscr{W}_k^r(\mathscr{K}^n)$

4) Further relations between the $\mathscr{W}_k^r(\mathscr{K})$ are given in [4].

The following two propositions are direct consequences of Propositions 4 and 2 of [11].

**Proposition 1.** *Either $\mathscr{W}_k^r(\mathscr{K}^n)$ is empty, or each component has codimension at most $r\cdot(d_n+r-k)$ in $V$.*

**Proposition 2.** *Assume $r>0$. Then the points of $\mathscr{W}_k^{r+1}(\mathscr{K}^n)$ are singular points of $\mathscr{W}_k^r(\mathscr{K}^n)$.*

**Proposition 3.** $\pi(\mathscr{W}_k^r(\mathscr{K}^n))$, if nonempty, is a closed complex subspace of $T_g$ of codimension at most $r\cdot(d_n+r-k)-1$.

*Proof.* The first assertion follows since $\pi$ is proper ([8]). The second assertion is a consequence of Proposition 1 and the fact that every compact Riemann surface has only finitely many $n$-fold Weierstrass points.

Note that the set $A_k^n=\pi(\mathscr{W}_k^1(\mathscr{K}^n)-\mathscr{W}_{k-1}^1(\mathscr{K}^n))$ consists of all $s\in T_g$ such that $V_s$ has an $n$-fold Weierstrass point with first non-$n$-gap $k$. Now, we have

$$\pi(\mathscr{W}_k^1(\mathscr{K}^n))-\pi(\mathscr{W}_{k-1}^1(\mathscr{K}^n))\subseteq A_k^n\subseteq\pi(\mathscr{W}_k^1(\mathscr{K}^n)).$$

Note that the first of these (whose points are all $s\in T_g$ such that $V_s$ has an $n$-fold Weierstrass point with first non-$n$-gap $k$ and no $n$-fold Weierstrass point with first non-$n$-gap less than $k$) is an open complex subspace of the last and that the dimension of the last, if nonempty, is at least $k+3g-d_n+3$. So if the first of these sets is nonempty, then we may say that "the Riemann surfaces of genus $g$ which have an $n$-fold Weierstrass point with first non-$n$-gap $k$ depend on at least $k+3g-d_n+3$ moduli." (If $n=1$, one can delete the words "at least" from this statement; cf. [1], Theorem 1 of [22], and Corollary 2 of [12].)

**Theorem 2.** *Let $\mathscr{W}_\infty(\mathscr{K})$ denote the topological space $\bigcup_{n\geqq 1}\mathscr{W}_{d_n}^1(\mathscr{K}^n)$. Then $\mathscr{W}_\infty(\mathscr{K})$ is dense in $V$.*

*Proof.* This follows easily from [19, Theorem 2] (also see [17, p. 11]).

## 3. Remarks on Tangent Space Dimension

In [12], we used a variational formula for abelian differentials, similar to formulas derived by Rauch [22], to study the tangent space at a point of $\mathscr{W}_k^1(\mathscr{K})$. We obtained the following result.

**Theorem 3.** *For $2\leqq k\leqq g$, $\mathscr{W}_k^1(\mathscr{K})-\mathscr{W}_k^2(\mathscr{K})$ is smooth of pure dimension $k+2g-3$.*

A similar analysis is possible at a point $(s,P)\in\mathscr{W}_k^1(\mathscr{K}^n)$ if $V_s$ is not hyperelliptic.

**Theorem 4** (M. Noether). *Suppose $X$ is a compact Riemann surface of genus greater than 2 and that $X$ is not hyperelliptic. Let $K$ denote the canonical bundle on $X$. The canonical map*

$$S^*H^0(X,K)\to\bigoplus_{n\geqq 1}H^0(X,K^n),$$

*where $S^*$ denotes symmetric algebra, is surjective.*

*Proof.* [23, 15].

Now suppose $V_{s_0}$ is not hyperelliptic and let $\omega_1,\ldots,\omega_g$ be a basis of holomorphic 1-forms on $V_{s_0}$. By Noether's theorem, there exist homogeneous

polynomials $P_1, \ldots, P_{d_n}$ of degree $n$ in $g$ variables such that $\{P_j(\omega_1, \ldots, \omega_g)\}_{j=1}^{d_n}$ is a basis of $H^0(V_{s_0}, K_{s_0}^n)$. Let $\omega_1^*(s), \ldots, \omega_g^*(s)$ be sections of $\pi_* \mathscr{K}$ such that $\omega_i^*(s_0) = \omega_i$, $i=1, 2, \ldots, g$ (cf. [12, 16]). Then

**Theorem 5.** *There exists an open neighborhood $U$ of $s_0 \in T_g$ such that*

$$\{P_j(\omega_1^*(s), \ldots, \omega_g^*(s))\}_{j=1}^{d_n}$$

*is a basis of $H^0(V_s, K_s^n)$ for all $s \in U$.*

Proof. This follows from a Wronskian argument as in [2].

Now, the variational formula (Proposition 4 of [12]) allows one to explicitly express the $P_j(\omega_1^*, \ldots, \omega_g^*)$ in terms of local coordinates at a point $(s_0, P) \in V$. Theorem 5 allows us to represent the map $u_k^n$ in a neighborhood of $(s_0, P)$ by a matrix, as in §2, of partial derivatives of the $P_j(\omega_1^*, \ldots, \omega_g^*)$. One may then attempt to perform a similar analysis to that in [12] to compute the dimension of the tangent space to a point of $\mathscr{W}_k^1(\mathscr{K}^n) - \mathscr{W}_k^2(\mathscr{K}^n)$. However, the situation becomes much more complicated when $n > 1$, and, in fact, a result such as Theorem 3 is no longer possible, as the following propositions show.

**Proposition 4.** *Suppose $g=2$. Then $|\mathscr{W}_2^1(\mathscr{K})| = |\mathscr{W}_3^1(\mathscr{K}^2)| = |\mathscr{W}_2^1(\mathscr{K}^2)|$.*

*Proof.* The first equality has been noted by Hubbard [10, p. 124]. To complete the proof, suppose $X = V_s$ is a compact Riemann surface of genus 2 and $P$ is a (hyperelliptic) Weierstrass point on $X$. Then there is a basis $\{\omega_1, \omega_2\}$ of $H^0(X, K)$ such that $\operatorname{ord}_P \omega_1 = 0$ and $\operatorname{ord}_P \omega_2 = 2$ (cf. [9]). Then $\{\omega_1^2, \omega_1 \omega_2, \omega_2^2\}$ is a linearly independent set of elements of $H^0(X, K^2)$, hence a basis. The sequence of 2-gaps at $P$ is then 1, 3, 5 and it is easy to see that $(s, P) \in \mathscr{W}_2^1(\mathscr{K}^2)$.

Thus, for $g=2$, $\dim \mathscr{W}_2^1(\mathscr{K}^2) = 3$ $(= \dim \mathscr{W}_2^1(\mathscr{K}))$, which exceeds the lower bound on dimension given by Proposition 1. Note that in this special case, even through we are dealing with hyperelliptic curves, the dimension of the tangent spaces can be computed by the procedure described above. If we let $T_{(s,P)}(\mathscr{W}_k^r(\mathscr{K}^n))$ denote the tangent space to $\mathscr{W}_k^r(\mathscr{K}^n)$ at $(s, P)$, then we have

**Proposition 5.** *Suppose $g=2$ and $(s_0, P) \in \mathscr{W}_2^1(\mathscr{K})$. Then*

(i) $\dim T_{(s_0,P)}(\mathscr{W}_2^1(\mathscr{K}^2)) = 3$

(ii) $\dim T_{(s_0,P)}(\mathscr{W}_3^1(\mathscr{K}^2)) = 4$.

*In particular, $\mathscr{W}_2^1(\mathscr{K}^2)$ is smooth and $\mathscr{W}_3^1(\mathscr{K}^2)$ is not reduced.*

*Proof.* Let $\{\omega_1, \omega_2\}$ be a basis of $H^0(V_{s_0}, K_{s_0})$ such that $\operatorname{ord}_P \omega_1 = 0$ and $\operatorname{ord}_P \omega_2 = 2$. Let $t$ denote a local coordinate on $V_{s_0}$ at $P$. Then we may write

$$\begin{aligned} \omega_1 &= \sum_{l=0}^{\infty} a_l t^l \, dt \qquad a_0 \neq 0 \\ \omega_2 &= \sum_{l=0}^{\infty} b_l t^l \, dt \qquad b_0 = b_1 = 0, \ b_2 \neq 0. \end{aligned} \tag{1}$$

Let $c_1, c_2, c_3$ denote Patt's local coordinates on $T_g$ at $s_0$ (cf. [20, 5]). Then $t, c_1, c_2, c_3$ are local coordinates on $V$ at $(s_0, P)$ [5, Theorem 1]. Letting $\omega_1^*, \omega_2^*$

denote the sections of $\pi_* \mathscr{K}$ as in [12] such that $\omega_1^*(s_0)=\omega_1$ and $\omega_2^*(s_0)=\omega_2$, we have that

$$\begin{aligned} \omega_1^* &= \sum_{l=0}^{\infty} a_l^* t^l \, dt \\ \omega_2^* &= \sum_{l=0}^{\infty} b_l^* t^l \, dt \end{aligned} \tag{2}$$

where

$$\begin{aligned} a_l^* &= a_l + \sum_{m=1}^{3} c_m \tau'_{P,l}(Q_m)\, \omega_1(Q_m) + O(c^2) \\ b_l^* &= b_l + \sum_{m=1}^{3} c_m \tau'_{P,l}(Q_m)\, \omega_2(Q_m) + O(c^2) \end{aligned} \tag{3}$$

with $\tau_{P,l}$ an elementary integral of the second kind on $V_{s_0}$ with pole of order $l+1$ at $P$ and $Q_1$, $Q_2$, $Q_3$ suitably chosen points on $V_{s_0}$ (see [12] for details).

Let $f_j(t, c_1, c_2, c_3)$, $j=1, 2, 3$, be defined by

$$\begin{aligned} (\omega_1^*)^2 &= f_1 \, dt^2 \\ \omega_1^* \omega_2^* &= f_2 \, dt^2 \\ (\omega_2^*)^2 &= f_3 \, dt^2. \end{aligned} \tag{4}$$

Then in a neighborhood of $(s_0, P)$, the map $u_3^2$ is represented by the matrix

$$[\partial^i f_j / \partial t^i] \qquad i=0, 1, 2;\ j=1, 2, 3.$$

Let $\xi$ denote a tangent vector to $V$ at $(s_0, P)$. We may view $\xi$ as a $\mathbb{C}$-homomorphism of local rings $\xi\colon \mathcal{O}_{V,(s_0,P)} \to \mathbb{C}[\varepsilon]/(\varepsilon^2)$. Now, $\xi$ is determined by its values on $t$, $c_1$, $c_2$, $c_3$. Let $u$, $v_1$, $v_2$, $v_3$ in $\mathbb{C}$ be defined by

$$\begin{aligned} \xi(t) &= u\varepsilon \\ \xi(c_m) &= v_m \varepsilon \qquad m=1, 2, 3. \end{aligned}$$

By [11, Proposition 3], $\xi \in T_{(s_0,P)}(\mathscr{W}_3^1(\mathscr{K}^2))$ if and only if

$$\det[\xi(\partial^i f_j / \partial t^i)] = 0 \qquad i=0, 1, 2;\ j=1, 2, 3. \tag{5}$$

Now, by Taylor's theorem we have

$$\begin{aligned} \xi(\partial^i f_j / \partial t^i) = {} & \partial^i f_j / \partial t^i (s_0, P) + u\varepsilon\, \partial^{i+1} f_j / \partial t^{i+1}(s_0, P) \\ & + \varepsilon \sum_{m=1}^{3} v_m \partial^{i+1} f_j / \partial c_m \partial t^i (s_0, P). \end{aligned}$$

Computing the entries of the matrix in (5) by using formulas (1)–(4) and the fact that $t$, $c_1$, $c_2$, $c_3$ all vanish at $(s_0, P)$, we find that $\xi(f_3) = \xi(\partial f_3/\partial t) = 0$ and that $\xi(f_2)$, $\xi(\partial f_2/\partial t)$, and $\xi(\partial^2 f_3/\partial t^2)$ all belong to the maximal ideal of $\mathbb{C}[\varepsilon]/(\varepsilon^2)$. Thus formula (5) is satisfied (since $\varepsilon^2=0$). Hence every tangent vector to $V$ at $(s_0, P)$ is a tangent vector to $\mathscr{W}_3^1(\mathscr{K}^2)$, establishing (ii).

Finally, for $\xi$ to be tangent to $\mathscr{W}_2^1(\mathscr{K}^2)$, it is necessary and sufficient that

$$[\xi(\partial^i f_j / \partial t^i)] \qquad i=0, 1;\ j=1, 2, 3$$

have rank 1. Since the last column of this matrix consists of zeros, this matrix has rank 1 exactly when

$$\det \begin{bmatrix} \xi(f_1) & \xi(f_2) \\ \xi(\partial f_1/\partial t) & \xi(\partial f_2/\partial t) \end{bmatrix} = 0.$$

It is easy to see that this imposes a nontrivial linear relation on $u, v_1, v_2, v_3$. Hence $T_{(s_0, P)}(\mathscr{W}_2^1(\mathscr{K}^2))$ is a three-dimensional subspace of $T_{(s_0, P)}(V)$.

Note that while a generic point of $\mathscr{W}_k^1(\mathscr{K}) - \mathscr{W}_k^2(\mathscr{K})$, for any genus, has $k$ as its first nongap (cf. [1, proof of Theorem 3.1]), no point of $\mathscr{W}_3^1(\mathscr{K}^2)$, for $g=2$, has 3 as its first non-2-gap. Indeed, each point of $\mathscr{W}_3^1(\mathscr{K}^2)$ is a 2-fold Weierstrass point of weight 3. It is then not surpising that $\mathscr{W}_3^1(\mathscr{K}^2)$ fails to be reduced.

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