JOURNAL OF ALGEBRA 178, 807–832 (1995)

On Canonical Ideals, Intersection Numbers, and Weierstrass Points on Gorenstein Curves

Arnaldo Garcia*

I.M.P.A., Estrada Dona Castorina 110, 22.460 Rio de Janeiro, Brazil

and

R. F. Lax†

Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana 70803

Communicated by D. A. Buchsbaum

Received March 30, 1994

1. INTRODUCTION

The investigation of Gorenstein curve singularities has been carried on by several authors. For example, Gorenstein [7] proved that all singularities of plane curves are of this type, and Roquette [15] generalized this result, giving to it a ring-theoretic treatment. Rosenlicht [16] (see also Stöhr [17]) proved a Riemann–Roch Theorem for curves with singularities, with the dualizing sheaf being locally free exactly when all singularities are of this type. Kunz [10] proved that, for unibranch singularities, the Gorenstein condition is equivalent to the symmetry of the semigroup of values at the singularity. This symmetry property of the semigroup of values has also been discussed by Waldi [18] and the first author [4] in the case of plane curves with two branches, and by Delgado [2, 3] and Stöhr [17] in the general case. For plane curves, Gorenstein [7] proved a formula relating the degree of the singularity at a point with the singularity degrees of the branches and the intersection numbers of each branch with the union of the other branches. Hironaka [9] generalized this formula to arbitrary curve singularities.

* Partially supported by a CNPq grant. E-mail: garcia@impa.br.
† Partially supported by NSF Grant INT-9101337. E-mail: lax@math.lsu.edu.

0021-8693/95 $12.00

Here we are mainly concerned with Gorenstein curve singularities. We note (cf. [5, Example (3.10)]) that the branches at such a singularity need not be Gorenstein. We obtain formulas (see (3.9) and (4.4)) that are improvements of those of Gorenstein and Hironaka. The central result (Theorem (3.3)) of this work is the determination of a "canonical ideal" for a subset of branches of the singularity. We obtain a formula (see (3.7)) relating the conductor ideal at the singularity with the conductors of the branches. This generalizes a result of Gorenstein [7] and is applied (see (3.13)) to obtain a formula for the conductor of the semigroup of values that generalizes known formulas in the cases of plane curves [7] and of singularities with precisely two branches [5, Proposition (3.6)]. We also prove "distributive properties" (see (4.1) and (4.3)) for intersection numbers of subsets of branches and give examples of families of Gorenstein singularities that illustrate the theorems on intersection numbers.

In the final section, we compute the Weierstrass weight of a singularity with an arbitrary number of branches on an integral, projective Gorenstein curve over an algebraically closed field of characteristic zero. This generalizes the formulas in [5] for the Weierstrass weight of singularities with one or two branches. The main theorem in the final section can be proved independently from the rest of the paper, but is closely related to the material in Section 3.

2. PRELIMINARIES

Let X be a projective curve (reduced scheme of pure dimension one) over an arbitrary base field. Let $\mathcal{O}$ denote the completion of the local ring of X at a singular point P, where P is a rational point over the base field. As follows from [16, Theorem, 1] and [8, proof of Satz 3.6], we have in our case that $\mathcal{O}$ is a reduced ring.

Let

$$\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_r$$

denote the minimal prime ideals of the local ring $\mathcal{O}$, with the prime ideal $\mathfrak{P}_j$ corresponding to the branch (i.e., an irreducible algebroid curve over the base field) X_j of X at P for $j = 1, 2, \ldots, r$. We remark that since the base field is not assumed to be algebraically closed, the branch X_j need not be absolutely irreducible.

We need to introduce some notation. Given a partition $S_1, S_2, \ldots, S_t$ of $\{1, 2, \ldots, r\}$, let

$$\bigcap_{k<j} \mathfrak{P}_k$$

denote the intersection of those minimal primes of $\mathscr{O}$ with index belonging to $S_1 \cup S_2 \cup \cdots \cup S_{j-1}$, and let

$$\bigcap_{k>j} \mathfrak{P}_k$$

denote the intersection of those minimal primes of $\mathscr{O}$ with index belonging to $S_{j+1} \cup \cdots \cup S_t$. Put

$$\bigcap_{k<1} \mathfrak{P}_k = \bigcap_{k>t} \mathfrak{P}_k = \mathscr{O}.$$

For a subset S of $\{1,2,\dots,r\}$, let S' denote the complementary set $\{1,2,\dots,r\} \setminus S$. We write

$$\bigcap_{k \notin S} \mathfrak{P}_k$$

for $\bigcap_{k \in S'} \mathfrak{P}_k$. Put

$$\mathscr{O}_S = \mathscr{O} \Big/ \bigcap_{k \in S} \mathfrak{P}_k.$$

When $S = \{j\}$, we write $\bigcap_{k \neq j} \mathfrak{P}_k$ instead of $\bigcap_{k \notin S} \mathfrak{P}_k$, and $\mathscr{O}_j$ instead of $\mathscr{O}_S$. Note that $\mathscr{O}_{\{1,2,\dots,r\}} = \mathscr{O}$. Also, we denote by

$$X_S = \bigcup_{k \in S} X_k$$

the union of the branches corresponding to indices in S.

For a ring R, we denote by $\tilde{R}$ its integral closure in its total ring of fractions. Also, given two rings $R_1 \subseteq R_2$, we put

$$(R_1 : R_2) \doteq \{\alpha \in R_1 \mid \alpha R_2 \subseteq R_1\}.$$

Let $\mathscr{F} = (\mathscr{O} : \tilde{\mathscr{O}})$ and $\mathscr{F}_S = (\mathscr{O}_S : \tilde{\mathscr{O}}_S)$, for a subset S of $\{1,2,\dots,r\}$, denote the respective conductor ideals. If $S = \{j\}$, we will write $\mathscr{F}_j$ instead of $\mathscr{F}_S$.

Throughout this work "dimension" refers to vector space dimension over the base field. The number $\delta = \dim \tilde{\mathscr{O}}/\mathscr{O}$ is called the *singularity degree of P over the base field.* The singularity degree may increase under base field extensions, but if X is totally rational (see [9, p. 182]) at P over the base field, then it is independent of such extensions. For a subset $S \subseteq \{1,2,\dots,r\}$, we put

$$\delta_S = \dim \tilde{\mathscr{O}}_S/\mathscr{O}_S,$$

while if $S = \{j\}$, we will write δ_j instead of δ_S.

If S_1 and S_2 are two disjoint, nonempty subsets of $\{1, 2, \ldots, r\}$, then one defines the intersection number

$$(X_{S_1} \bullet X_{S_2}) = \dim \frac{\mathcal{O}}{\left(\bigcap_{j \in S_1} \mathfrak{P}_j\right) + \left(\bigcap_{k \in S_2} \mathfrak{P}_k\right)}.$$

This intersection number does not change under base field extensions (see [9, p. 180]).

(2.1) DEFINITION. Let R be a ring with total ring of fractions K. We will call an R-module $\mathfrak{A} \subseteq K$ a *fractional ideal* of R if there is a non-zerodivisor s of R such that $s\mathfrak{A} \subseteq R$ and, moreover, if $\mathfrak{A}$ contains some non-zerodivisor of R (i.e., if $\mathfrak{A} \cdot K = K$).

(2.2) DEFINITION [8]. A fractional ideal $\mathfrak{C}$ of R is called *R-canonical* if for every fractional ideal $\mathfrak{A}$ of R, we have the following reciprocity property:

$$\mathfrak{A} = (\mathfrak{C} : (\mathfrak{C} : \mathfrak{A})).$$

We also refer to $\mathfrak{C}$ as a canonical ideal for the ring R. It follows from [8, Satz 2.8] that if a canonical ideal exists, then it is unique up to multiplication by fractional principal ideals.

If $\mathfrak{A}$ and $\mathfrak{B}$ are any two fractional ideals of R, then we put

$$(\mathfrak{A} : \mathfrak{B}) = \{\alpha \in K \mid \alpha \cdot \mathfrak{B} \subseteq \mathfrak{A}\}.$$

(2.3) PROPOSITION. *Let R be a one-dimensional Cohen–Macaulay ring, and let $\tilde{R}$ be the integral closure of R in its total ring of fractions. Put $\mathcal{F} = (R : \tilde{R})$, and let $\mathfrak{C}$ be a fractional ideal of R.*

(1) *$\mathfrak{C}$ is R-canonical if and only if for every two fractional ideals $\mathfrak{A}$ and $\mathfrak{B}$ with $\mathfrak{A} \subseteq \mathfrak{B}$, we have*

$$\text{length}\, \mathfrak{B}/\mathfrak{A} = \text{length}(\mathfrak{C} : \mathfrak{A})/(\mathfrak{C} : \mathfrak{B}).$$

(2) *$\mathfrak{C}$ is R-canonical if and only if for every two fractional ideals $\mathfrak{A}$ and $\mathfrak{B}$, we have*

$$(\mathfrak{A} : \mathfrak{B}) = (\mathfrak{C} : \mathfrak{B}) : (\mathfrak{C} : \mathfrak{A}).$$

(3) *If $\mathfrak{C}$ is R-canonical, then $(\mathfrak{C} : \mathfrak{C}) = R$ and $(\mathfrak{C} : \mathfrak{C} \cdot \tilde{R}) = \mathfrak{F}$.*

Proof. (1) If $\mathfrak{C}$ is R-canonical, then the mapping $\mathfrak{A} \mapsto (\mathfrak{C} : \mathfrak{A})$ is a bijection on the set of fractional ideals of R that reverses inclusions. From this, the "only if" part follows easily. We now show the "if" part. Let $\mathfrak{A}$ be a fractional ideal of R. Obviously, $\mathfrak{A} \subseteq (\mathfrak{C} : (\mathfrak{C} : \mathfrak{A}))$. Hence we have by assumption

$$\operatorname{length}(\mathfrak{C} : (\mathfrak{C} : \mathfrak{A}))/\mathfrak{A} = \operatorname{length}(\mathfrak{C} : \mathfrak{A})/(\mathfrak{C} : (\mathfrak{C} : (\mathfrak{C} : \mathfrak{A}))).$$

But, it is easy to see that

$$(\mathfrak{C} : (\mathfrak{C} : (\mathfrak{C} : \mathfrak{A}))) = (\mathfrak{C} : \mathfrak{A}).$$

Therefore, $\mathfrak{A} = (\mathfrak{C} : (\mathfrak{C} : \mathfrak{A}))$. (This argument is essentially the same as the one given in [15, Satz 4]; also see [17].)

(2) For any fractional ideal $\mathfrak{C}$, we have

$$(\mathfrak{C} : \mathfrak{B}) : (\mathfrak{C} : \mathfrak{A}) = (\mathfrak{C} : \mathfrak{B} \cdot (\mathfrak{C} : \mathfrak{A})) = (\mathfrak{C} : (\mathfrak{C} : \mathfrak{A})) : \mathfrak{B}.$$

If $\mathfrak{C}$ is R-canonical, then we have $(\mathfrak{A} : \mathfrak{B}) = (\mathfrak{C} : \mathfrak{B}) : (\mathfrak{C} : \mathfrak{A})$. For the converse statement, put $\mathfrak{B} = R$.

(3) Since $\mathfrak{C}$ is R-canonical, we have $(\mathfrak{C} : (\mathfrak{C} : \mathfrak{A})) = \mathfrak{A}$. Taking $\mathfrak{A} = R$, we get $(\mathfrak{C} : \mathfrak{C}) = R$. Now,

$$(\mathfrak{C} : \mathfrak{C} \cdot \tilde{R}) = (\mathfrak{C} : \mathfrak{C}) : \tilde{R} = (R : \tilde{R}) = \mathfrak{F}. \quad \blacksquare$$

Recall that the local ring $\mathcal{O}$ and the singularity at P are said to be Gorenstein if

$$\dim \mathcal{O}/\mathcal{F} = \dim \tilde{\mathcal{O}}/\mathcal{O}.$$

(2.4) THEOREM. *Let R be a one-dimensional Cohen–Macaulay local ring. Then R is R-canonical if and only if R is Gorenstein.*

Proof. See [8, pp. 27–29]. $\blacksquare$

(2.5) LEMMA. *Let R be a one-dimensional Cohen–Macaulay ring. If $\mathfrak{C}$ is R-canonical and if $\tilde{R}$ is a principal ideal ring, then*

$$(\mathfrak{C} : \mathfrak{C} \cdot \mathcal{F}) = \tilde{R} \qquad \text{and} \qquad (\mathfrak{C} : \tilde{R}) = \mathfrak{C} \cdot \mathcal{F}.$$

Proof. This follows from Proposition (2.3) (3) using that $\mathfrak{F}$ and $\mathfrak{C} \cdot \tilde{R}$ are principal fractional ideals of $\tilde{R}$. $\blacksquare$

3. CANONICAL AND CONDUCTOR IDEALS

Given a partition $S_1, S_2, \ldots, S_t$ of $\{1, 2, \ldots, r\}$, we have a commutative diagram

$$\begin{array}{ccccccccc}
\tilde{\mathcal{O}} & \xrightarrow{\cong} & \tilde{\mathcal{O}}_{S_1} & \times & \tilde{\mathcal{O}}_{S_2} & \times & \cdots & \times & \tilde{\mathcal{O}}_{S_t} \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
\mathcal{O} & \xrightarrow{\psi} & \mathcal{O}_{S_1} & \times & \mathcal{O}_{S_2} & \times & \cdots & \times & \mathcal{O}_{S_t},
\end{array}$$

where ψ is the diagonal morphism. Since we have that $\mathcal{O}$ is reduced, the map ψ is injective, and we will identify $\mathcal{O}$ with its isomorphic image $\psi(\mathcal{O})$. Put

$$\mathcal{O}' = \mathcal{O}_{S_1} \times \mathcal{O}_{S_2} \times \cdots \times \mathcal{O}_{S_t}.$$

(3.1) PROPOSITION. *Let $S_1, S_2, \ldots, S_t$ be a partition of $\{1, 2, \ldots, r\}$. Then, with notation as in the preceding section, we have*

$$\begin{aligned}\delta &= \sum_{j=1}^{t} \delta_{S_j} + \sum_{j=1}^{t} \left(X_{S_j} \bullet \bigcup_{k<j} X_{S_k} \right) \\ &= \sum_{j=1}^{t} \delta_{S_j} + \sum_{j=1}^{t} \left(X_{S_j} \bullet \bigcup_{k>j} X_{S_k} \right).\end{aligned}$$

Proof. From [9, Proposition 1], we have

$$\dim \mathcal{O}'/\mathcal{O} = \sum_{j=1}^{t} \left(X_{S_j} \bullet \bigcup_{k<j} X_{S_k} \right).$$

As Hironaka points out [9, p. 181], it follows that this sum of intersection numbers is independent of permutations of the sets S_j. The proposition now follows from the diagram above. ∎

Let S_1 and S_2 be disjoint subsets of $\{1, \ldots, r\}$. Then we note that

$$\left(\bigcap_{k \in S_1} \mathfrak{P}_k \right) \cdot \mathcal{O}_{S_2}$$

is a fractional ideal (in our sense) of $\mathcal{O}_{S_2}$, i.e., it contains some non-zero-divisor of $\mathcal{O}_{S_2}$. This follows from the fact that

$$\bigcap_{k \in S_1} \mathfrak{P}_k \not\subseteq \bigcup_{k \in S_2} \mathfrak{P}_k.$$

(3.2) PROPOSITION. *For $j = 1, 2, \ldots, t$, let $\mathfrak{A}_j$ be a fractional ideal of $\mathcal{O}_{S_j}$ and put*

$$\mathfrak{A} = \mathfrak{A}_1 \times \mathfrak{A}_2 \times \cdots \times \mathfrak{A}_t.$$

Then we have for any fractional ideal J contained in $\mathcal{O}$

$$(J : \mathfrak{A}) = \prod_{j \geq 1} \left(\left(\bigcap_{k \notin S_j} \mathfrak{P}_k \cap J \right) \cdot \mathcal{O}_{S_j} : \mathfrak{A}_j \right).$$

In particular,

$$(J:\mathcal{O}') = \prod_{j\geq 1}\left(\bigcap_{k\notin S_j}\mathfrak{P}_k \cap J\right)\cdot \mathcal{O}_{S_j}.$$

Proof. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_t)$, with α_j in the total ring of fractions of $\mathcal{O}_{S_j}$, be such that

$$\alpha_j \cdot \mathfrak{A}_j \subseteq \left(\bigcap_{k\notin S_j}\mathfrak{P}_k \cap J\right)\cdot \mathcal{O}_{S_j},$$

and take $\beta = (\beta_1, \beta_2, \dots, \beta_t) \in \mathfrak{A} = \mathfrak{A}_1 \times \mathfrak{A}_2 \times \cdots \times \mathfrak{A}_t$. Then $\alpha \cdot \beta = (\alpha_1\beta_1, \alpha_2\beta_2, \dots, \alpha_t\beta_t)$ is such that

$$\alpha_j \cdot \beta_j \in \left(\bigcap_{k\notin S_j}\mathfrak{P}_k \cap J\right)\cdot \mathcal{O}_{S_j}.$$

For each j, take $g_j \in \bigcap_{k\notin S_j}\mathfrak{P}_k \cap J$ a preimage of $\alpha_j \cdot \beta_j$, and put $g = g_1 + g_2 + \cdots + g_t$. We then have $g \in J$ and $\psi(g) = (\bar{g}, \bar{g}, \dots, \bar{g}) = (\bar{g}_1, \bar{g}_2, \dots, \bar{g}_t) = (\alpha_1\beta_1, \dots, \alpha_t\beta_t)$. This shows the inclusion

$$(J:\mathfrak{A}) \supseteq \prod_{j\geq 1}\left(\left(\bigcap_{k\notin S_j}\mathfrak{P}_k \cap J\right)\cdot \mathcal{O}_{S_j} : \mathfrak{A}_j\right)$$

For the opposite inclusion, take an element $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_t) \in (J:\mathfrak{A})$. Then, for any $\beta_1 \in \mathfrak{A}_1$, one has $\alpha \cdot (\beta_1, 0, \dots, 0) = (\alpha_1\beta_1, 0, \dots, 0) \in J$. Since $J \subseteq \mathcal{O}$ we have that there exists $g \in J$ with $(\alpha_1\beta_1, 0, \dots, 0) = \psi(g) = (\bar{g}, \bar{g}, \dots, \bar{g})$. This means that

$$\alpha_1\beta_1 \in \left(\bigcap_{k\notin S_1}\mathfrak{P}_k \cap J\right)\cdot \mathcal{O}_{S_1},$$

and hence $\alpha_1 \in ((\bigcap_{k\notin S_1}\mathfrak{P}_k \cap J)\cdot \mathcal{O}_{S_1} : \mathfrak{A}_1)$. The same argument applies to the other coordinates of $(\alpha_1, \dots, \alpha_t)$ and finishes the proof. ∎

The next result plays a key role in this paper. It is the determination of a canonical ideal for the ring $\mathcal{O}/I$, where I is the ideal of a set of branches. (See [11, Lemma 2.1] for the determination of a canonical module for the case when $\sqrt{I}$ is the maximal ideal.)

(3.3) THEOREM. *Let $\mathfrak{C} \subseteq \mathcal{O}$ be a canonical ideal for the ring $\mathcal{O}$ and let S be a subset of $\{1, \dots, r\}$. Then $(\bigcap_{k\notin S}\mathfrak{P}_k \cap \mathfrak{C})\cdot \mathcal{O}_S$ is a canonical ideal for the ring $\mathcal{O}_S$.*

Proof. Let $\mathfrak{A} \subseteq \mathfrak{B}$ be two fractional ideals of $\mathcal{O}_S$. Put

$$\mathfrak{A}' = \mathfrak{A} \times \prod_{j \notin S} \mathcal{O}_j, \qquad \mathfrak{B}' = \mathfrak{B} \times \prod_{j \notin S} \mathcal{O}_j.$$

Then $\dim \mathfrak{B}'/\mathfrak{A}' = \dim \mathfrak{B}/\mathfrak{A}$.

Since $\mathfrak{C}$ is $\mathcal{O}$-canonical, we have

$$\dim \mathfrak{B}'/\mathfrak{A}' = \dim(\mathfrak{C} : \mathfrak{A}')/(\mathfrak{C} : \mathfrak{B}').$$

From Proposition (3.2), since $\mathfrak{C} \subseteq \mathcal{O}$ by hypothesis, we have

$$\dim \frac{(\mathfrak{C} : \mathfrak{A}')}{(\mathfrak{C} : \mathfrak{B}')} = \dim \frac{\left(\left(\bigcap_{k \notin S} \mathfrak{P}_k \cap \mathfrak{C}\right) \cdot \mathcal{O}_S : \mathfrak{A}\right)}{\left(\left(\bigcap_{k \notin S} \mathfrak{P}_k \cap \mathfrak{C}\right) \cdot \mathcal{O}_S : \mathfrak{B}\right)}.$$

The theorem then follows from (1) of Proposition (2.3). ∎

(3.4) COROLLARY. *Let $\mathfrak{C} \subseteq \mathcal{O}$ be a canonical ideal for the ring $\mathcal{O}$, and let S be a subset of $\{1, \dots, r\}$. Then*

$$\delta_S = \dim \frac{(\bigcap_{k \notin S} \mathfrak{P}_k \cap \mathscr{C}) \cdot \mathcal{O}_S}{(\bigcap_{k \notin S} \mathfrak{P}_k \cap \mathfrak{C}) \cdot \mathscr{F}_S}.$$

Proof. Put $\mathfrak{C}_S = (\bigcap_{k \notin S} \mathfrak{P}_k \cap \mathfrak{C}) \cdot \mathcal{O}_S$. Since $\mathfrak{C}_S$ is $\mathcal{O}_S$-canonical, we have

$$\dim \frac{\mathfrak{C}_S}{\mathfrak{C}_S \cdot \mathscr{F}_S} = \dim \frac{(\mathfrak{C}_S : \mathfrak{C}_S \cdot \mathscr{F}_S)}{(\mathfrak{C}_S : \mathfrak{C}_S)} = \dim \frac{(\mathfrak{C}_S : \mathfrak{C}_S \cdot \mathscr{F}_S)}{\mathcal{O}_S}.$$

But $(\mathfrak{C}_S : \mathfrak{C}_S \cdot \mathscr{F}_S) = \tilde{\mathcal{O}}_S$ from Lemma (2.5). ∎

(3.5) COROLLARY. *Let $\mathfrak{C} \subseteq \mathcal{O}$ be a canonical ideal for the ring $\mathcal{O}$ and let S be a subset of $\{1, \dots, r\}$. Then $\mathcal{O}_S$ is Gorenstein if and only if $(\bigcap_{k \notin S} \mathfrak{P}_k \cap \mathfrak{C}) \cdot \mathcal{O}_S$ is a principal ideal.*

Proof. This follows from Theorem (3.3) and Theorem (2.4). ∎

(3.6) PROPOSITION. *Let $\mathfrak{C}$ be a canonical ideal for the ring $\mathcal{O}$ (not necessarily contained in $\mathcal{O}$), and let $\mathfrak{A}$ be a fractional $\mathcal{O}$-ideal. Then we have*

$$(\mathfrak{C} : \tilde{\mathcal{O}}) = (\mathfrak{C} : \mathfrak{A}) \cdot (\mathfrak{A} : \tilde{\mathcal{O}}).$$

Proof. For any fractional ideal $\mathfrak{A}$ for the ring $\mathcal{O}$, we have that $(\mathfrak{A} : \tilde{\mathcal{O}})$ is an $\tilde{\mathcal{O}}$-ideal and hence $\tilde{\mathcal{O}}$-principal. Therefore,

$$\tilde{\mathcal{O}} = \mathfrak{A} : (\mathfrak{A} : \tilde{\mathcal{O}}).$$

Since $\mathfrak{C}$ is $\mathscr{O}$-canonical, we get

$$\tilde{\mathscr{O}} = (\mathfrak{C} : (\mathfrak{C} : \mathfrak{A})) : (\mathfrak{A} : \tilde{\mathscr{O}}) = \mathfrak{C} : (\mathfrak{C} : \mathfrak{A}) \cdot (\mathfrak{A} : \tilde{\mathscr{O}}).$$

Applying the operator $(\mathfrak{C} : \ )$ on both sides, we get

$$(\mathfrak{C} : \tilde{\mathscr{O}}) = \mathfrak{C} : \big(\mathfrak{C} : (\mathfrak{C} : \mathfrak{A}) \cdot (\mathfrak{A} : \tilde{\mathscr{O}})\big) = (\mathfrak{C} : \mathfrak{A}) \cdot (\mathfrak{A} : \tilde{\mathscr{O}}). \quad \blacksquare$$

The following theorem generalizes a result of Gorenstein [7, Theorem 2] in the case of plane curves.

(3.7) THEOREM. *Let $\mathfrak{C} \subseteq \mathscr{O}$ be a canonical ideal for the ring $\mathscr{O}$, and let $S_1, S_2, \ldots, S_t$ be a partition of $\{1, \ldots, r\}$. Then we have*

$$(\mathscr{C} : \tilde{\mathscr{O}}) = \prod_{j \geq 1} \left(\bigcap_{k \notin S_j} \mathfrak{P}_k \cap \mathfrak{C} \right) \cdot \mathscr{F}_{S_j}.$$

In particular, if $\mathscr{O}$ is a Gorenstein ring, we have

$$\mathscr{F} = \prod_{j \geq 1} \left(\bigcap_{k \notin S_j} \mathfrak{P}_k \right) \cdot \mathscr{F}_{S_j}.$$

Proof. Put $\mathscr{O}' = \mathscr{O}_{S_1} \times \mathscr{O}_{S_2} \times \cdots \times \mathscr{O}_{S_t}$. Then, obviously, we have

$$(\mathscr{O}' : \tilde{\mathscr{O}}) = \mathscr{F}_{S_1} \times \mathscr{F}_{S_2} \times \cdots \times \mathscr{F}_{S_t}.$$

The theorem then follows from Proposition (3.6) (taking $\mathfrak{A} = \mathscr{O}'$) and from Proposition (3.2) (taking $J = \mathfrak{C}$). $\blacksquare$

(3.8) *Remark.* Theorem (3.7) also follows from Proposition (3.2) (taking $\mathfrak{A} = \tilde{\mathscr{O}}$) and Lemma (2.5).

(3.9) THEOREM. *Suppose $\mathscr{O}$ is a Gorenstein ring and let $S_1, S_2, \ldots, S_t$ be a partition of $\{1, \ldots, r\}$. Then the singularity degree (over the base field) can be given by*

$$\delta = \sum_{j=1}^{t} \delta_{S_j} + \frac{1}{2} \sum_{j=1}^{t} (X_{S_j} \bullet X_{S'_j}),$$

where S'_j denotes the complementary set of S_j in $\{1, \ldots, r\}$.

Proof. As before, put

$$\mathscr{O}' = \prod_{j=1}^{t} \mathscr{O}_{S_j}.$$

Then

$$\delta = \dim \frac{\tilde{\mathcal{O}}}{\mathcal{O}} = \dim \frac{\tilde{\mathcal{O}}}{\mathcal{O}'} + \dim \frac{\mathcal{O}'}{\mathcal{O}} = \sum_{j=1}^{t} \delta_{S_j} + \dim \frac{\mathcal{O}'}{\mathcal{O}}.$$

Now, since $\mathcal{O}$ is Gorenstein, we have

$$\dim \frac{\mathcal{O}'}{\mathcal{O}} = \dim \frac{(\mathcal{O} : \mathcal{O})}{(\mathcal{O} : \mathcal{O}')} = \dim \frac{\mathcal{O}}{(\mathcal{O} : \mathcal{O}')}.$$

Therefore,

$$\dim \frac{\mathcal{O}'}{(\mathcal{O} : \mathcal{O}')} = \dim \frac{\mathcal{O}'}{\mathcal{O}} + \dim \frac{\mathcal{O}}{(\mathcal{O} : \mathcal{O}')} = 2 \dim \frac{\mathcal{O}'}{\mathcal{O}}.$$

Hence, we have

$$\begin{aligned}\delta &= \sum_{j=1}^{t} \delta_{S_j} + \frac{1}{2} \dim \frac{\mathcal{O}'}{(\mathcal{O} : \mathcal{O}')} \\ &= \sum_{j=1}^{t} \delta_{S_j} + \frac{1}{2} \sum_{j=1}^{t} (X_{S_j} \bullet X_{S_j'}),\end{aligned}$$

using Proposition (3.2) with $J = \mathcal{O}$. ∎

For a subset S of $\{1, \ldots, r\}$, put

$$\mathcal{I}_S = (X_S \bullet X_{S'}).$$

If $S = \{j\}$, we will write $\mathcal{I}_j$ instead of $\mathcal{I}_S$. With this notation, Theorem (3.9) says that

$$\delta = \sum_{j=1}^{t} \left(\delta_{S_j} + \frac{1}{2} \mathcal{I}_{S_j} \right)$$

in the case of a Gorenstein singularity.

(3.10) Corollary. *Suppose $\mathcal{O}$ is a Gorenstein ring and let S be a subset of $\{1, \ldots, r\}$. Let $T_1, T_2, \ldots, T_s$ be a partition of the set S. Then we have*

$$\delta_S + \frac{1}{2} \mathcal{I}_S = \sum_{j=1}^{s} \delta_{T_j} + \frac{1}{2} \sum_{j=1}^{s} \mathcal{I}_{T_j}.$$

In particular, we have

$$\mathcal{I}_S \leq \sum_{j=1}^{s} \mathcal{I}_{T_j}.$$

Proof. Apply Theorem (3.9) to the partition S, S' and to the partition $T_1, T_2, \dots, T_s, S'$. The last assertion follows from the fact that

$$\delta_S \geq \sum_{j=1}^{s} \delta_{T_j}. \quad \blacksquare$$

(3.11) DEFINITION. For a curve singularity (not necessarily Gorenstein) we call

$$\mathfrak{J} = \left(\delta - \sum_{j=1}^{r} \delta_j\right)$$

the *intersection degree* of the singularity.

It follows from [9] and Proposition (3.1) that this number does not change under base field extensions.

(3.12) COROLLARY. *Suppose $\mathscr{O}$ is a Gorenstein ring. Then the intersection degree of the singularity is given by*

$$\mathfrak{J} = \frac{1}{2}\sum_{j=1}^{r} \mathscr{I}_j.$$

Proof. This follows from Theorem (3.9) by taking $S_j = \{j\}$ for $j = 1, \dots, r$. $\blacksquare$

Assuming that $\mathscr{O}$ is Gorenstein, we have, from Corollary (3.12), that the following equality holds:

$$\dim \frac{\tilde{\mathscr{O}}}{\mathscr{F}} = 2\delta = \sum_{j=1}^{r} \mathscr{I}_j + 2\sum_{j=1}^{r} \delta_j.$$

The next theorem gives a better description of this equality.

Let ν_j denote the valuation corresponding to the valuation ring $\tilde{\mathscr{O}}_j$, for $j = 1, 2, \dots, r$. The *semigroup of values $\mathscr{H}$ of the singularity* is defined by

$$\mathscr{H} = \{(\nu_1(f), \nu_2(f), \dots, \nu_r(f)) \mid f \in \mathscr{O}, f \text{ a non-zerodivisor}\}.$$

Then $\mathscr{H}$ is a subsemigroup of the semigroup $\mathbf{N}^r$ of r-tuples of natural numbers. The conductor of $\mathscr{H}$, denoted $c(\mathscr{H})$, is the least element of $\mathscr{H}$ (with respect to the product ordering of $\mathbf{N}^r$) such that

$$c(\mathscr{H}) + \mathbf{N}^r \subseteq \mathscr{H}.$$

These semigroups have been studied by, among others, Delgado [2, 3] and the first author [4]. The following proposition generalizes a result of

Gorenstein [7] in the case of plane curves and [5, Proposition (3.6)] in the two-branch case.

(3.13) THEOREM. *Suppose $\mathscr{O}$ is a Gorenstein ring. Then*

$$c(\mathscr{H}) = (\mathscr{I}_1 + 2\delta_1, \mathscr{I}_2 + 2\delta_2, \dots, \mathscr{I}_r + 2\delta_r).$$

Proof. Take $S_j = \{j\}$ for $j = 1, 2, \dots, r$. It then follows from Theorem (3.7), taking $\mathfrak{C} = \mathscr{O}$, that $c(\mathscr{H}) = (\xi_1, \xi_2, \dots, \xi_r)$, where

$$\xi_j = \dim \frac{\tilde{\mathscr{O}}_j}{\left(\bigcap_{k \neq j} \mathfrak{P}_k\right) \cdot \mathscr{F}_j}$$

$$= \dim \frac{\tilde{\mathscr{O}}_j}{\mathscr{O}_j} + \dim \frac{\mathscr{O}_j}{\left(\bigcap_{k \neq j} \mathfrak{P}_k\right) \cdot \mathscr{O}_j} + \dim \frac{\left(\bigcap_{k \neq j} \mathfrak{P}_k\right) \cdot \mathscr{O}_j}{\left(\bigcap_{k \neq j} \mathfrak{P}_k\right) \cdot \mathscr{F}_j}$$

$$= \delta_j + \mathscr{I}_j + \delta_j,$$

using Corollary (3.4) with $S = \{j\}$. ∎

4. INTERSECTION NUMBERS AT A GORENSTEIN SINGULARITY

If S_1 and S_2 are two disjoint, nonempty subsets of $\{1, 2, \dots, r\}$, put

$$\mathscr{I}_{S_1, S_2} = (X_{S_1} \bullet X_{S_2}).$$

If $S_2 = \{k\}$, we will write $\mathscr{I}_{S_1, k}$ for $\mathscr{I}_{S_1, S_2}$, while if $S_1 = \{j\}$ and $S_2 = \{k\}$, we will write $\mathscr{I}_{j,k}$ for $\mathscr{I}_{S_1, S_2}$. Note that $\mathscr{I}_S = \mathscr{I}_{S, S'}$.

Let $S_1, S_2, \dots, S_t$ be a partition of $\{1, 2, \dots, r\}$ and take j such that $1 \leq j \leq t$. Put

$$A_j = S_1 \cup S_2 \cup \cdots \cup S_{j-1}$$
$$B_j = S_{j+1} \cup S_{j+2} \cup \cdots \cup S_t.$$

By comparing Proposition (3.1) with Theorem (3.9), we see that if $\mathscr{O}$ is Gorenstein, then

$$\sum_{j=1}^{t} \mathscr{I}_{S_j} = \sum_{j=1}^{t} \left(\mathscr{I}_{S_j, A_j} + \mathscr{I}_{S_j, B_j}\right).$$

The next result shows that corresponding terms in these two sums are equal.

(4.1) THEOREM (Distributive Property). *Suppose $\mathcal{O}$ is a Gorenstein ring and let S be a subset of $\{1,\ldots,r\}$. Let S_1, S_2 be a partition of the complementary set $S' = \{1,\ldots,r\}\setminus S$. Then we have*

$$\mathcal{I}_S = \mathcal{I}_{S,S'} = \mathcal{I}_{S,S_1} + \mathcal{I}_{S,S_2}.$$

Proof. Put

$$\mathcal{O}' = \mathcal{O}_S \times \mathcal{O}_{S_1} \times \mathcal{O}_{S_2}.$$

Ordering the subsets S, S_1, S_2 as (S_1, S, S_2), we see from [9, Proposition 1] that

$$\dim\frac{\mathcal{O}'}{\mathcal{O}} = \mathcal{I}_{S,S_1} + \mathcal{I}_{S_2}.$$

On the other hand, ordering these sets as (S_2, S, S_1), we see that

$$\dim\frac{\mathcal{O}'}{\mathcal{O}} = \mathcal{I}_{S,S_2} + \mathcal{I}_{S_1}.$$

Now, using the fact that $\mathcal{O}$ is Gorenstein, we have, as in the proof of Theorem (3.9),

$$\dim\frac{\mathcal{O}'}{\mathcal{O}} = \frac{1}{2}\dim\frac{\mathcal{O}'}{(\mathcal{O}:\mathcal{O}')} = \frac{1}{2}(\mathcal{I}_S + \mathcal{I}_{S_1} + \mathcal{I}_{S_2}).$$

Hence, we have

$$\mathcal{I}_S + \mathcal{I}_{S_1} + \mathcal{I}_{S_2} = (\mathcal{I}_{S,S_1} + \mathcal{I}_{S_2}) + (\mathcal{I}_{S,S_2} + \mathcal{I}_{S_1}),$$

and the theorem is proved. ∎

(4.2) COROLLARY. *Suppose $\mathcal{O}$ is a Gorenstein ring and let S, S_1, S_2 be a partition of $\{1,\ldots,r\}$. Then we have*

$$\left(\left(\bigcap_{k\notin S}\mathfrak{P}_k\right)\cdot\mathcal{O}_S : \left(\bigcap_{k\in S_1}\mathfrak{P}_k\right)\cdot\mathcal{O}_S\right) = \left(\bigcap_{k\in S_2}\mathfrak{P}_k\right)\cdot\mathcal{O}_S.$$

Proof. Since $\mathcal{O}$ is Gorenstein, we have from Theorem (3.3) that $(\bigcap_{k\notin S}\mathfrak{P}_k)\cdot\mathcal{O}_S$ is $\mathcal{O}_S$-canonical. Hence,

$$\mathcal{I}_{S,S_1} = \dim\frac{\mathcal{O}_S}{\left(\bigcap_{k\in S_1}\mathfrak{P}_k\right)\cdot\mathcal{O}_S} = \dim\frac{\left(\left(\bigcap_{k\notin S}\mathfrak{P}_k\right)\cdot\mathcal{O}_S : \left(\bigcap_{k\in S_1}\mathfrak{P}_k\right)\cdot\mathcal{O}_S\right)}{\left(\bigcap_{k\notin S}\mathfrak{P}_k\right)\cdot\mathcal{O}_S}.$$

But, from Theorem (4.1), we also have

$$\begin{aligned}\mathscr{I}_{S,S_1} &= \mathscr{I}_S - \mathscr{I}_{S,S_2}\\ &= \dim\frac{\mathscr{O}_S}{\left(\bigcap_{k\notin S}\mathfrak{P}_k\right)\cdot\mathscr{O}_S} - \dim\frac{\mathscr{O}_S}{\left(\bigcap_{k\in S_2}\mathfrak{P}_k\right)\cdot\mathscr{O}_S}\\ &= \dim\frac{\left(\bigcap_{k\in S_2}\mathfrak{P}_k\right)\cdot\mathscr{O}_S}{\left(\bigcap_{k\notin S}\mathfrak{P}_k\right)\cdot\mathscr{O}_S}.\end{aligned}$$

Since we obviously have

$$\left(\bigcap_{k\in S_2}\mathfrak{P}_k\right)\cdot\mathscr{O}_S \subseteq \left(\left(\bigcap_{k\notin S}\mathfrak{P}_k\right)\cdot\mathscr{O}_S : \left(\bigcap_{k\in S_1}\mathfrak{P}_k\right)\cdot\mathscr{O}_S\right),$$

it follows that these ideals are equal. ∎

As has been pointed out by Hironaka [9, p. 181] and Buchweitz and Greuel [1, p. 247] by using the example of the three coordinate axes in three-space, which is not a Gorenstein singularity, one does not have a "*strong distributive property*" for the intersection numbers in general; i.e., it is *not* true in general that $\mathscr{I}_j = \sum_{k\neq j}\mathscr{I}_{j,k}$, or, in different notation, it is not true that

$$\left(X_j \bullet \bigcup_{k\neq j} X_k\right) = \sum_{k\neq j}(X_j \bullet X_k).$$

Notice that Theorem (4.1) above shows that this strong distributive property holds in the case of a Gorenstein singularity with three branches. This property also holds if X is a plane curve or if P is a smooth point of a surface that contains all the X_j's [9, p. 181]. The next theorem gives a more general hypothesis on the singularity such that this distributive property holds. In the following section, we give examples to show that this property fails if the number of branches is at least four and one only assumes that $\mathscr{O}$ is Gorenstein.

(4.3) THEOREM. *Let S be a subset of $\{1,\ldots,r\}$ and let S' denote the complementary set. Suppose that $\mathscr{O}$ and $\mathscr{O}_{S'}$ are both Gorenstein rings. Then,*

$$\mathscr{I}_S = \sum_{k\notin S}\mathscr{I}_{S,k}.$$

In particular, if the hypotheses are satisfied for $S = \{j\}$, then

$$\mathcal{I}_j = \sum_{k \neq j} \mathcal{I}_{j,k}.$$

Proof. Since $\mathcal{O}$ is Gorenstein,

$$\Big(\bigcap_{k \in S} \mathfrak{P}_k\Big) \cdot \mathcal{O}_{S'} = \Big(\bigcap_{k \notin S'} \mathfrak{P}_k\Big) \cdot \mathcal{O}_{S'}$$

is $\mathcal{O}_{S'}$-canonical by Theorem (3.3). Now, since $\mathcal{O}_{S'}$ is also assumed to be Gorenstein, $(\bigcap_{k \in S} \mathfrak{P}_k) \cdot \mathcal{O}_{S'}$ is a principal ideal, say

$$\Big(\bigcap_{k \in S} \mathfrak{P}_k\Big) \cdot \mathcal{O}_{S'} = g_S \cdot \mathcal{O}_{S'},$$

for some $g_S \in (\bigcap_{k \in S} \mathfrak{P}_k)$. It follows that if $j \in S'$, then

$$\Big(\bigcap_{k \in S} \mathfrak{P}_k\Big) \cdot \mathcal{O}_j = g_S \cdot \mathcal{O}_j;$$

i.e., $(\bigcap_{k \in S} \mathfrak{P}_k) + \mathfrak{P}_j = g_S \cdot \mathcal{O} + \mathfrak{P}_j$ if $j \notin S$.

Then,

$$\begin{aligned}
\mathcal{I}_S &= \dim \frac{\mathcal{O}_{S'}}{\Big(\bigcap_{k \in S} \mathfrak{P}_k\Big) \cdot \mathcal{O}_{S'}} \\
&= \dim \frac{\mathcal{O}_{S'}}{g_S \cdot \mathcal{O}_{S'}} = \dim \frac{\tilde{\mathcal{O}}_{S'}}{g_S \cdot \tilde{\mathcal{O}}_{S'}} \\
&= \sum_{j \notin S} \dim \frac{\tilde{\mathcal{O}}_j}{g_S \cdot \tilde{\mathcal{O}}_j} = \sum_{j \notin S} \dim \frac{\mathcal{O}_j}{g_S \cdot \mathcal{O}_j} \\
&= \sum_{j \notin S} \dim \frac{\mathcal{O}}{\mathfrak{P}_j + g_S \cdot \mathcal{O}} = \sum_{j \notin S} \dim \frac{\mathcal{O}}{\Big(\bigcap_{k \in S} \mathfrak{P}_k\Big) + \mathfrak{P}_j} \\
&= \sum_{j \notin S} \mathcal{I}_{S,j}. \quad \blacksquare
\end{aligned}$$

The next theorem generalizes a result of Gorenstein [7, p. 428] in the case of plane curves (see also Hironaka [9, Proposition 2]).

(4.4) THEOREM. *Suppose $\mathscr{O}$ is a Gorenstein ring. Suppose moreover that, for each $1 \leq j \leq r$, the ring $\mathscr{O}/\bigcap_{k \neq j} \mathfrak{P}_k$ is also Gorenstein. Then we have*

$$\delta = \sum_{j=1}^{r} \delta_j + \sum_{1 \leq i < k \leq r} \mathscr{I}_{i,k}.$$

Proof. Let $S_j = \{j\}$ for $j = 1, 2, \ldots, r$. The theorem then follows from Theorems (3.9) and (4.3). ∎

5. EXAMPLES

In this section we construct families of Gorenstein local subrings $\mathscr{O}$ of finite codimension in the ring

$$k[[t]]^r = k[[t]] \times \cdots \times k[[t]] \qquad (r \text{ copies}),$$

where k is a field. For some of these rings we calculate intersection numbers of branches and embedding dimensions. In particular, we show that Theorems (4.3) and (4.4) do not hold in general for Gorenstein singularities. We also give geometric realizations for some of these rings. We thank Professor W.-D. Geyer for the geometric realization in the first family below.

We denote elements of $k[[t]]^r$ by $(A_1, A_2, \ldots, A_r)$, and elements of k^r by $(\alpha_1, \alpha_2, \ldots, \alpha_r)$. We will always assume that $\alpha_i \neq 0$ for all $1 \leq i \leq r$; i.e., $(\alpha_1, \alpha_2, \ldots, \alpha_r)$ will actually denote an element in $(k^*)^r$. The power series A_i will be written as

$$A_i = \sum_{j \geq 0} a_j^{(i)} t^j.$$

Fix an element $(\alpha_1, \alpha_2, \ldots, \alpha_r) \in (k^*)^r$ and consider the subspace $\mathscr{O}$ of $k[[t]]^r$ of codimension r given as below:

$$a_0^{(1)} = a_0^{(2)} = \cdots = a_0^{(r)}$$
$$\alpha_1 a_1^{(1)} + \alpha_2 a_1^{(2)} + \cdots + \alpha_r a_1^{(r)} = 0. \qquad (5.1)$$

One easily checks that $\mathscr{O}$ is a local ring with integral closure $\tilde{\mathscr{O}}$ equal to $k[[t]]^r$. Also, the conductor ideal is

$$(\mathscr{O} : \tilde{\mathscr{O}}) = (t^2)^r = (t^2) \times \cdots \times (t^2).$$

Hence $\mathscr{O}$ is Gorenstein. It will follow from the computations in the next family of examples (5.2) that here one has the following values for intersection numbers:

(1) $\mathcal{I}_{i,S} = 1$, for $1 \le i \le r$ and $S \subsetneqq \{1,\dots,r\} \setminus \{i\}$

(2) $\mathcal{I}_i = 2$, for $1 \le i \le r$.

For r even and $\alpha_i = (-1)^i$, $1 \le i \le r$, the embedding dimension of $\mathcal{O}$ is equal to $(r-1)$. With $r = 4$ and $\alpha_i = (-1)^i$, a minimal set of generators of the maximal ideal of $\mathcal{O}$ is $(0,0,t,t)$, $(t,t,0,0)$, and $(t,-t,-t,t)$. The ring

$$k[[X,Y,Z]]/(X^2+Y^2-Z^2, XY)$$

is then a geometrical realization of $\mathcal{O}$ (so the singularity here is a complete intersection). More precisely, the following homomorphism is injective and has image the ring $\mathcal{O}$ (with $r = 4$ and $\alpha_i = (-1)^i$):

$$k[[X,Y,Z]]/(X^2+Y^2-Z^2, XY) \to k[[t]] \times k[[t]] \times k[[t] \times k[[t]]$$

$$f(X,Y,Z) \mapsto (f(0,t,t), f(0,t,-t),$$

$$f(t,0,-t), f(t,0,t)).$$

For our second family of examples, let $n \ge 1$ be a natural number and fix an element $(\alpha_1, \alpha_2, \dots, \alpha_r) \in (k^*)^r$ satisfying

$$\sum_{i=1}^{r} \alpha_i = 0.$$

Let $\mathcal{O}$ be the subspace of $k[[t]]^r$ of codimension $r(n+1)$ given as below:

$$A_1 \equiv A_2 \equiv \cdots \equiv A_r \mod t^{n+1}$$

$$\sum_{i=1}^{r} \alpha_i A_i \equiv 0 \mod t^{2n+2} \tag{5.2}$$

The hypothesis $\sum_{i=1}^r \alpha_i = 0$ is crucial here in order to prove that $\mathcal{O}$ is a ring; i.e., the subspace $\mathcal{O}$ is closed under multiplication in $k[[t]]^r$. One easily checks that $\mathcal{O}$ is local with maximal ideal $\mathcal{M}$ given by

$$\mathcal{M} = \{(A_1, A_2, A_r) \in \mathcal{O} \mid A_1 \equiv 0 \mod t\}.$$

Also, the integral closure is $\tilde{\mathcal{O}} = k[[t]]^r$ and the conductor ideal is

$$(\mathcal{O} : \tilde{\mathcal{O}}) = (t^{2n+2})^r = (t^{2n+2}) \times (t^{2n+2}) \times \cdots \times (t^{2n+2}).$$

Hence $\dim \tilde{\mathcal{O}}/(\mathcal{O} : \tilde{\mathcal{O}}) = 2r(n+1)$ and $\mathcal{O}$ is a Gorenstein ring.

The minimal prime ideals of $\mathcal{O}$ are given by

$$\mathfrak{P}_i = \{(A_1, A_2, \ldots, A_r) \in \mathcal{O} \mid A_i = 0\}, \qquad \text{for } 1 \leq i \leq r.$$

Regarding intersection numbers, one has here

(1) $\mathcal{I}_{i,S} = (n+1)$, for $1 \leq i \leq r$ and $S \subsetneq \{1, \ldots, r\} \setminus \{i\}$

(2) $\mathcal{I}_i = 2(n+1)$, for $1 \leq i \leq r$.

We will now establish the above formulas for $i = 1$. They follow from the equalities below:

$$(1) \quad \mathfrak{P}_1 + \mathfrak{P}_2 = \mathfrak{P}_1 + (\mathfrak{P}_2 \cap \mathfrak{P}_3) = \cdots = \mathfrak{P}_1 + (\mathfrak{P}_2 \cap \cdots \cap \mathfrak{P}_{r-1})$$
$$= \{(A_1, \ldots, A_r) \in \mathcal{O} \mid A_1 \equiv 0 \bmod t^{n+1}\}$$

$$(2) \quad \mathfrak{P}_1 + (\mathfrak{P}_2 \cap \cdots \cap \mathfrak{P}_r) = \{(A_1, \ldots, A_r) \in \mathcal{O} \mid A_1 \equiv 0 \bmod t^{2n+2}\}.$$

We will prove the first equality above, the second being easier.

We clearly have

$$\mathfrak{P}_1 + (\mathfrak{P}_2 \cap \cdots \cap \mathfrak{P}_{r-1}) \subseteq \cdots \subseteq \mathfrak{P}_1 + (\mathfrak{P}_2 \cap \mathfrak{P}_3) \subseteq \mathfrak{P}_1 + \mathfrak{P}_2$$
$$\subseteq \{(A_1, \ldots, A_r) \in \mathcal{O} \mid A_1 \equiv 0 \bmod t^{n+1}\}.$$

We just need to show then the inclusion

$$\{(A_1, \ldots, A_r) \in \mathcal{O} \mid A_1 \equiv 0 \bmod t^{n+1}\} \subseteq \mathfrak{P}_1 + (\mathfrak{P}_2 \cap \cdots \cap \mathfrak{P}_{r-1}).$$

We will write elements of $\mathfrak{P}_1$ as $(0, B_2, \ldots, B_r)$ and elements of $\mathfrak{P}_2 \cap \cdots \cap \mathfrak{P}_{r-1}$ as $(C_1, 0, \ldots, 0, C_r)$. Let $(A_1, \ldots, A_r) \in \mathcal{O}$ with $A_1 \equiv 0 \bmod t^{n+1}$. Take

$$C_1 = A_1, \qquad B_2 = A_2, \qquad B_3 = A_3, \ldots, B_{r-1} = A_{r-1}.$$

The desired result then reduces to choosing B_r and C_r such that $B_r + C_r = A_r$ and the elements $(0, B_2, \ldots, B_r)$ and $(C_1, 0, \ldots, 0, C_r)$ both belong to the ring $\mathcal{O}$. Let

$$\mathcal{T}_s\left(\sum_{j \geq 0} a_j t^j\right) = \sum_{j \leq s} a_j t^j$$

denote the s-truncation of a power series. One choice for such elements B_r and C_r is then

$$B_r = -\frac{1}{\alpha_r} \cdot \mathcal{T}_{2n+1}\left(\sum_{i=2}^{r-1} \alpha_i A_i\right)$$

$$C_r = -\frac{\alpha_1}{\alpha_r} \cdot \mathcal{T}_{2n+1}(A_1) + A_r - \mathcal{T}_{2n+1}(A_r).$$

For r even and $\alpha_i = (-1)^i$, $1 \leq i \leq r$, the embedding dimension of $\mathscr{O}$ is equal to r. Put $T = t^{n+1}$. A minimal set of generators of the maximal ideal of $\mathscr{O}$ for $r = 4$ is given by $(0,0,T,T)$, $(T,T,0,0)$, $(T,-T,-T,T)$, and (t,t,t,t). The ring

$$R = k[[X,Y,Z,W]]/(X^2 + Y^2 - Z^2, XY, X + Y - W^{n+1})$$

realizes $\mathscr{O}$ geometrically. More precisely, the following homomorphism is injective and has as its image the ring $\mathscr{O}$ (with $r = 4$ and $\alpha_i = (-1)^i$):

$$\begin{aligned} R &\to k[[t]] \times k[[t]] \times k[[t]] \times k[[t]] \\ f(X,Y,Z,W) &\mapsto (f(0,T,T,t), f(0,T,-T,t), \\ &\qquad f(T,0,-T,t), f(T,0,T,t)). \end{aligned}$$

For our third family, let $n \geq 0$ and $m \geq 0$ be natural numbers and let r be even. Let $\mathscr{O}$ denote the subspace of $k[[t]]^r$ of codimension $r(n+1) + (rm/2)$ given as below:

$$\begin{aligned} &A_1 \equiv A_2 \equiv \cdots \equiv A_r \mod t^{n+1} \\ &A_1 \equiv A_2; A_3 \equiv A_4; \cdots; A_{r-1} \equiv A_r \mod t^{n+m+1} \\ &\sum_{i=1}^{r} (-1)^i A_i \equiv 0 \mod t^{2n+m+2}. \end{aligned} \tag{5.3}$$

As before, one can check that $\mathscr{O}$ is a local ring having integral closure $\bar{\mathscr{O}} = k[[t]]^r$. Since the conductor ideal is

$$(\mathscr{O} : \bar{\mathscr{O}}) = (t^{2n+2+m})^r = (t^{2n+2+m}) \times \cdots \times (t^{2n+2+m}),$$

we have that $\mathscr{O}$ is a Gorenstein ring. Regarding intersection numbers, one has here

(1) $\mathscr{I}_{1,S} = n + 1$, if $S \subseteq \{3,4,\ldots,r\}$

(2) $\mathscr{I}_{1,S} = n + m + 1$, if $2 \in S \subsetneqq \{2,3,\ldots,r\}$

(3) $\mathscr{I}_1 = 2n + 2 + m$.

The values of these intersection numbers are obtained using the following assertions, whose proofs are similar to the proof given above involving the preceding family of rings.

$$\begin{aligned} (1) \quad \mathfrak{P}_1 + \mathfrak{P}_3 &= \mathfrak{P}_1 + (\mathfrak{P}_3 \cap \mathfrak{P}_4) = \cdots = \mathfrak{P}_1 + (\mathfrak{P}_3 \cap \cdots \cap \mathfrak{P}_r) \\ &= \{(A_1,\ldots,A_r) \in \mathscr{O} \mid A_1 \equiv 0 \mod t^{n+1}\} \end{aligned}$$

(2) $\mathfrak{P}_1 + \mathfrak{P}_2 = \mathfrak{P}_1 + (\mathfrak{P}_2 \cap \mathfrak{P}_3) = \cdots = \mathfrak{P}_1 + (\mathfrak{P}_2 \cap \cdots \cap \mathfrak{P}_{r-1})$

$$= \{(A_1, \ldots, A_r) \in \mathcal{O} \mid A_1 \equiv 0 \bmod t^{n+m+1}\}$$

(3) $\mathfrak{P}_1 + (\mathfrak{P}_2 \cap \cdots \cap \mathfrak{P}_r)$

$$= \{(A_1, \ldots, A_r) \in \mathcal{O} \mid A_1 \equiv 0 \bmod t^{2n+2+m}\}$$

(5.4) *Remark.* One could mix the examples in (5.2) and (5.3) to get other examples of Gorenstein rings $\mathcal{O}$ as below (where r is an even number):

$$A_1 \equiv A_2 \equiv \cdots \equiv A_r \mod t^{n+1}$$

$$A_1 \equiv A_2; A_3 \equiv A_4; \cdots; A_{r-1} \equiv A_r \mod t^{n+m+1}$$

$$\sum_{i=1}^{r} \alpha_i A_i \equiv 0 \mod t^{2n+2+m},$$

where $\alpha_1 + \alpha_2 = 0, \alpha_3 + \alpha_4 = 0, \ldots, \alpha_{r-1} + \alpha_r = 0$.

6. WEIERSTRASS WEIGHT OF A GORENSTEIN SINGULARITY

Now let X denote an integral, projective Gorenstein curve over an algebraically closed field k. Weierstrass points have been defined in the case of a line bundle on X when k is of characteristic 0 in [19, 20, 12], and in the case of a linear system on X in arbitrary characteristic in [6]. In order to arrive at a formula for the weight of a Gorenstein singularity that is not overly complicated, we will assume throughout this section that the characteristic of k is zero. One can modify this formula, taking into account the order sequences of the linear systems involved, and obtain an analogous result in positive characteristic. In particular, the formula for weight that we will derive gives a lower bound on the weight of a singularity on a Gorenstein canonical curve in positive characteristic if all linear systems involved have classical gap sequences.

Let $\omega = \omega_X$ denote the sheaf of dualizing differentials on X. Since X is Gorenstein, ω is locally free [16, Theorem 10]. Suppose $P \in X$ and let $\mathcal{O}$ denote the local ring of the structure sheaf of X at P. Let $\tau \in H^0(X, \omega)$ generate ω at P. Then, locally at P, τ is of the form dt/h, where t is a local coordinate at the points on the normalization of X that lie over P and h is a generator of the conductor of $\mathcal{O}$ in its integral closure.

We briefly recall some definitions that are more fully explained in [5, 6, 12, 19, 20]. The Weierstrass weight of P, denoted $W_X(P)$, is the order of vanishing at P of

$$\det\left[h^{i-1}d^{i-1}f_j/dt^{i-1}\right], \qquad i,j = 1,2,\ldots,g,$$

where $f_1\tau, f_2\tau, \ldots, f_g\tau$ are local representations at P for a basis for $H^0(X, \omega)$. More generally, if $\mathscr{L}$ is a line bundle on X and if $V \subset H^0(X, \mathscr{L})$ is a linear system of (affine) dimension s on X, then $W_V(P)$ is the order of vanishing at P of

$$\det\left[h^{i-1}d^{i-1}F_j/dt^{i-1}\right], \qquad i,j = 1,2,\ldots,s,$$

where the rational functions $F_1, F_2, \ldots, F_s$ are regular in a neighborhood of P and correspond to a basis for V.

In Section 3, we have defined the semigroup of values $\mathscr{H}$ of the singularity. We denote by $\mathscr{H}_k$ the projection of $\mathscr{H}$ on the kth coordinate. Then $\mathscr{H}_k$ is a numerical semigroup (i.e., $\mathbf{N} \setminus \mathscr{H}_k$ is a finite set). The elements of $\mathbf{N} \setminus \mathscr{H}_k$ are called gaps of $\mathscr{H}_k$.

(6.1) DEFINITION. If $\mathscr{S}$ is a numerical semigroup with gaps $l_1, l_2, \ldots, l_\delta$, then we define the *weight of* $\mathscr{S}$, denoted $\mathrm{wt}(\mathscr{S})$, by

$$\mathrm{wt}(\mathscr{S}) = \sum_{j=1}^{\delta} (l_j - j).$$

We define the weight of $\mathbf{N}$ to be 0.

(6.2) LEMMA. *If $\mathscr{S}$ is a numerical semigroup with δ gaps and if $0 = n_0, n_1, \ldots, n_{\delta-1}$ are the elements of $\mathscr{S}$ less than 2δ, then*

$$\sum_{i=0}^{\delta-1} (n_i - i) = (\delta - 1)\delta - \mathrm{wt}(\mathscr{S}).$$

Proof. See [5, Lemma (2.3)]. ∎

Let Y denote the partial normalization of X at P; i.e., Y is obtained by resolving the singularity at P, while leaving any other singularities unchanged. Let $Q_1, Q_2, \ldots, Q_r$ denote the points on Y that correspond to the r branches at P. The main result of this section is:

(6.3) THEOREM. *Let X denote an integral, projective Gorenstein curve of arithmetic genus $g > 1$. Let ω denote the dualizing bundle on X. Suppose P is a singular point of X with precisely r branches and singularity degree δ. For $k = 1,2,\ldots,r$, let $\mathscr{H}_k$ denote the semigroup of the kth branch, and let δ_k denote the number of gaps of $\mathscr{H}_k$. Let $\mathfrak{I} = \delta - \sum_{k=1}^{r}\delta_k$ be the intersection degree at P. Let Y denote the partial normalization of X at P, and let*

$Q_1, Q_2, \ldots, Q_r$ denote the points on Y that lie over P. For $k = 1, 2, \ldots, r$, let V_k denote the linear system on Y consisting of all rational differentials $\sigma \in H^0(X, \omega)$ such that σ is regular at Q_k on Y. Then

$$W_X(P) = \delta(g-1)(g+1) - \mathfrak{I}(g-1) - \sum_{k=1}^{r} \mathrm{wt}(\mathscr{H}_k) + \sum_{k=1}^{r} W_{V_k}(Q_k).$$

Proof. Let $\tau_1, \tau_2, \ldots, \tau_g$ be a basis for $H^0(X, \omega)$. Suppose $\tau \in H^0(X, \omega)$ generates ω at P. Locally at P, write $\tau_j = f_j \tau$ for $j = 1, 2, \ldots, g$. From [6, Proposition (1.6)], we have

$$W_X(P) = \delta(g-1)g + \sum_{k=1}^{r} \mathrm{ord}_{Q_k} \det\left[d^{i-1} f_j / dt^{i-1}\right],$$

where $i, j = 1, 2, \ldots, g$. Theorem (6.3) will follow from the next result.

(6.4) LEMMA. *For each $k = 1, 2, \ldots, r$,*

$$\mathrm{ord}_{Q_k} \det\left[d^{i-1} f_j / dt^{i-1}\right] = \delta_k(g-1) - \mathrm{wt}(\mathscr{H}_k) + W_{V_k}(Q_k).$$

Proof. We will prove the proposition for $k = 1$. Let $0 = m_0, m_1, \ldots, m_{\xi_1 - \delta_1}$ denote the first $\xi_1 - \delta_1 + 1$ elements of $\mathscr{H}_1$, where ξ_1 is the first coordinate of the conductor of $\mathscr{H}$. We have $\xi_1 = \mathscr{I}_1 + 2\delta_1$ by Theorem (3.13). Notice that $m_{\delta_1} = 2\delta_1$. Hence, $m_{\delta_1 + j} = 2\delta_1 + j$ for $j = 0, 1, \ldots, \xi_1 - 2\delta_1$.

For $m \in \mathbf{N}$, define an ideal $J(m)$ in $\mathscr{O}$ by

$$J(m) = \{f \in \mathscr{O} \mid \nu_1(f) \geq m\},$$

where ν_1 is the discrete valuation corresponding to Q_1. Consider the ideals $J_t \subseteq \mathscr{O}$, where for $t = 0, 1, \ldots, \xi_1 - \delta_1$,

$$J_t = J(m_t).$$

We have

$$\mathscr{O} = J_0 \supset J_1 \supset \cdots \supset J_{\xi_1 - \delta_1}.$$

Notice that if t is any integer greater than or equal to the conductor of $\mathscr{H}_1$, then $m_{t - \delta_1} = t$, since δ_1 is the number of gaps of $\mathscr{H}_1$. In particular, we have $m_{\xi_1 - \delta_1} = \xi_1$. Since the m_i are elements of $\mathscr{H}_1$, it is not hard to see that this chain of ideals is strictly decreasing; in fact, the length of each ideal is precisely one greater than the preceding ideal in the chain. In particular, we have

$$\dim_k \mathscr{O}/J_{\xi_1 - \delta_1} = \xi_1 - \delta_1.$$

Suppose $\tau \in H^0(X, \omega)$ generates ω at P. Then, from [5, Proposition (3.13)], there exist $\xi_1 - \delta_1$ linearly independent dualizing differentials $\tau_1, \tau_2, \dots, \tau_{\xi_1 - \delta_1} \in H^0(X, \omega)$ such that, locally at P, we have $\tau_j = f_j \tau$ with $f_j \in J_{j-1} \setminus J_j$ for $j = 1, 2, \dots, \xi_1 - \delta_1$. (Alternatively, one could show the existence of such differentials by using Corollary 2.14 of [17].) In particular, this means that

$$\nu_1(f_j) = m_{j-1} \qquad \text{for } j = 1, 2, \dots, \xi_1 - \delta_1.$$

We may extend these differentials to a basis $\tau_1, \tau_2, \dots, \tau_g$ for $H^0(X, \omega)$ in such a way that if $\tau_j = f_j \tau$ locally at P for $j = 1, \dots, g$, then $\nu_1(f_j) < \nu_1(f_{j+1})$ for $j = 1, \dots, g - 1$.

Notice that if $j > \xi_1 - \delta_1$, then $\nu_1(f_j) \geq \xi_1$. We may assume that $\tau = dt/h$, where t has order 1 at $Q_1, \dots, Q_r \in Y$ and h generates (in $\tilde{\mathcal{O}}$) the conductor of $\mathcal{O}$ in $\tilde{\mathcal{O}}$. Then we have $\nu_1(h) = \xi_1$. It follows that $\tau_1, \tau_2, \dots, \tau_{\xi_1 - \delta_1}$ have poles at Q_1 and that $\tau_{\xi_1 - \delta_1 + 1}, \dots, \tau_g$ are regular at Q_1 and may have poles at Q_k for $k = 2, \dots, r$. The linear system V_1 of rational differentials $\sigma \in H^0(X, \omega)$ such that σ is regular at Q_1 on Y is thus spanned by

$$\tau_{\xi_1 - \delta_1 + 1}, \dots, \tau_g.$$

Since Q_1 is a smooth point of Y, we have that dt generates the dualizing sheaf ω_Y of Y in a neighborhood of Q_1. Therefore, since $f_j \tau = (f_j/h)\, dt$, we have that $W_{V_1}(Q_1)$ is equal to the order of vanishing at Q_1 of the ordinary Wronskian (with respect to t) of the functions f_j/h for $j = \xi_1 - \delta_1 + 1, \dots, g$.

Now, we have

$$\begin{aligned}
&\operatorname{ord}_{Q_1} \det\left[d^{i-1} f_j / dt^{i-1}\right] \\
&= \sum_{j=1}^{g} \left[\nu_1(f_j) - (j-1)\right] \\
&= \sum_{j=1}^{\xi_1 - \delta_1} \left[m_{j-1} - (j-1)\right] + \sum_{j=\xi_1 - \delta_1 + 1}^{g} \left[\nu_1(f_j) - (j-1)\right] \\
&= \sum_{j=1}^{\delta_1} \left[m_{j-1} - (j-1)\right] + \sum_{j=\delta_1 + 1}^{\xi_1 - \delta_1} \left[m_{j-1} - (j-1)\right] \\
&\quad + \sum_{j=\xi_1 - \delta_1 + 1}^{g} \left[\nu_1(f_j/h) + \xi_1 - (j-1)\right].
\end{aligned}$$

From Lemma (6.2), it follows that

$$\begin{aligned}
&\operatorname{ord}_{Q_1}\det\left[d^{i-1}f_j/dt^{i-1}\right] \\
&\quad = (\delta_1 - 1)\delta_1 - \operatorname{wt}(\mathscr{H}_1) + \sum_{l=0}^{\xi_1 - 2\delta_1 - 1} [2\delta_1 + l - (\delta_1 + l)] \\
&\qquad + (g - \xi_1 + \delta_1)\xi_1 + \sum_{j=\xi_1-\delta_1+1}^{g} \left[\nu_1(f_j/h) - (j-1)\right] \\
&\quad = (\delta_1 - 1)\delta_1 - \operatorname{wt}(\mathscr{H}_1) + \delta_1(\xi_1 - 2\delta_1) \\
&\qquad + (g - \xi_1 + \delta_1)\xi_1 - (g - \xi_1 + \delta_1)(\xi_1 - \delta_1) \\
&\qquad + \sum_{s=1}^{g-\xi_1+\delta_1} \left[\nu_1(f_{\xi_1-\delta_1+s}/h) - (s-1)\right].
\end{aligned}$$

Since $W_{V_1}(Q_1)$ equals the last sum in the above equality, we finally have

$$\begin{aligned}
\operatorname{ord}_{Q_1}\det\left[d^{i-1}f_j/dt^{i-1}\right] &= (\delta_1 - 1)\delta_1 - \operatorname{wt}(\mathscr{H}_1) - \delta_1^2 + g\delta_1 + W_{V_1}(Q_1) \\
&= \delta_1(g-1) - \operatorname{wt}(\mathscr{H}_1) + W_{V_1}(Q_1). \quad \blacksquare
\end{aligned}$$

To conclude the proof of Theorem (6.3), notice that we have

$$\begin{aligned}
W_X(P) &= \delta(g-1)g + \sum_{k=1}^{r} \operatorname{ord}_{Q_k}\det\left(d^{i-1}f_j/dt^{i-1}\right) \\
&= \delta(g-1)g + \sum_{k=1}^{r} \delta_k(g-1) - \sum_{k=1}^{r} \operatorname{wt}(\mathscr{H}_k) + \sum_{k=1}^{r} W_{V_k}(Q_k) \\
&= (g-1)\left(\delta g + \sum_{k=1}^{r} \delta_k\right) - \sum_{k=1}^{r} \operatorname{wt}(\mathscr{H}_k) + \sum_{k=1}^{r} W_{V_k}(Q_k) \\
&= (g-1)(\delta g + \delta - \mathfrak{I}) - \sum_{k=1}^{r} \operatorname{wt}(\mathscr{H}_k) + \sum_{k=1}^{r} W_{V_k}(Q_k) \\
&= \delta(g-1)(g+1) - \mathfrak{I}(g-1) \\
&\qquad - \sum_{k=1}^{r} \operatorname{wt}(\mathscr{H}_k) + \sum_{k=1}^{r} W_{V_k}(Q_k). \quad \blacksquare
\end{aligned}$$

A careful analysis of the proofs of Lemma (6.4) and Theorem (6.3) shows that we did not need to use Theorem (3.1), which gave an explicit description for the conductor of $\mathscr{H}$. Therefore, this section is independent from the rest of the paper.

We call a singularity P *overweight* if, with the notation in (6.3), $W_{V_k}(Q_k) > 0$ for some k. The second author [13] has shown that a generic rational (irreducible) curve with g nodes has no overweight nodes. However, if the nodes occur in special position, then they may be overweight. The case of rational curves with three nodes was studied by the second author and Widland [14]. An example of a curve with overweight singularities is the plane curve $x^2y^2 + y^2z^2 = x^2z^2$. This curve has nodes at the points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, with each of these nodes having Weierstrass weight 8. (A node on a curve of arithmetic genus 3 is overweight if its weight is greater than 6.)

Using rational curves with three nodes, it is easy to construct a 1-parameter family $\{X_\lambda\}$ of curves with overweight singularities. Let X_λ denote the nodal curve obtained from the complex projective line by identifying 0 with ∞, 1 with -1, and i with λ, where $\lambda \notin \{0,1,-1,i,\infty\}$. Let P_λ denote the node formed by identifying i with λ. It follows from [14] that P_λ has weight 7 for $\lambda \neq -i$ and weight 8 for $\lambda = -i$. (The curve X_{-i} is the curve $x^2y^2 + y^2z^2 = x^2z^2$.)

As a consequence of Theorem (6.3) and the fact that the total of the Weierstrass weights equals $g^3 - g$, we get the following result for the number of smooth Weierstrass points on a Gorenstein curve with no overweight singularities.

(6.5) THEOREM. *Suppose X is an integral, projective Gorenstein curve of arithmetic genus g and geometric genus $\tilde{g}$. Let $P_1, P_2, \ldots, P_n$ denote the singularities of X and let r_i denote the number of branches at P_i for $i = 1,2,\ldots,n$. For each i, let $\mathscr{H}_{ij}$, $j = 1,2,\ldots,r_i$, denote the semigroup of the jth branch at P_i and let $\mathfrak{I}_i$ denote the intersection degree of P_i. Assume that no singularity of X is overweight. Then the number of smooth Weierstrass points on X, counting multiplicities, is*

$$\tilde{g}(g-1)(g+1) + \sum_{i=1}^{n}\left[(g-1)\mathfrak{I}_i + \sum_{j=1}^{r_i} \mathrm{wt}(\mathscr{H}_{ij})\right].$$

REFERENCES

1. R.-O. Buchweitz and G.-M. Greuel, The Milnor number and deformations of complex curve singularities, *Invent. Math.* **58** (1980), 241–281.
2. F. Delgado, The semigroup of values of a curve singularity with several branches, *Manuscripta Math.* **59** (1987), 347–374.
3. F. Delgado, Gorenstein curves and symmetry of the semigroup of values, *Manuscripta Math.* **61** (1988), 285–296.
4. A. Garcia, Semigroups associated to singular points of plane curves, *J. Reine Angew. Math.* **336** (1982), 165–184.

5. A. Garcia and R. F. Lax, Weierstrass weight of Gorenstein singularities with one or two branches, *Manuscripta Math.* **81** (1993), 361–378.
6. A. Garcia and R. F. Lax, Weierstrass points on Gorenstein curves in arbitrary characteristic, *Comm. Algebra* **22** (1994), 4841–4854.
7. D. Gorenstein, An arithmetic theory of adjoint plane curves, *Trans. Amer. Math. Soc.* **72** (1952), 414–436.
8. J. Herzog and E. Kunz, "Der kanonische Modul eines Cohen-Macaulay-Rings," Lecture Notes in Mathematics, Vol. 238, Springer, Berlin/Heidelberg/New York, 1971.
9. H. Hironaka, On the arithmetic genera and the effective genera of algebraic curves, *Mem. College Sci. Univ. Kyoto Ser. A* **30** (1957), 178–195.
10. E. Kunz, The value-semigroup of a one-dimensional Gorenstein ring, *Proc. Amer. Math. Soc.* **25** (1970), 748–751.
11. E. Kunz and R. Waldi, Über den Derivationenmodul und das Jacobi-Ideal von Kurvensingularitäten, *Math. Z.* **187** (1984), 105–123.
12. R. F. Lax, On the distribution of Weiertstrass points on singular curves, *Israel J. Math.* **57** (1987), 107–115.
13. R. F. Lax, Weierstrass points on rational nodal curves, *Glasgow Math. J.* **29** (1987), 131–140.
14. R. F. Lax and C. Widland, Weierstrass points on rational nodal curves of genus 3, *Canad. Math. Bull.* **30** (1987), 286–294.
15. P. Roquette, Über den Singularitätsgrad von teilringen in Funktionenkörpern, *Math. Z.* **77** (1961), 228–240.
16. M. Rosenlicht, Equivalence relations on algebraic curves, *Ann. of Math.* **56** (1952), 169–191.
17. K.-O. Stöhr, On the poles of regular differentials of singular curves, *Bol. Soc. Brasil Mat.* **24** (1993), 105–136.
18. R. Waldi, "Wertehalbgruppe und Singularität einer algebraischen Kurve," Dissertation, Regensburg, 1972.
19. C. Widland, "On Weierstrass points of Gorenstein curves," Ph.D. dissertation, Louisiana State University, 1984.
20. C. Widland and R. F. Lax, Weierstrass points on Gorenstein curves, *Pacific J. Math.* **142** (1990), 197–208.