1553 Spring 09 Test 3 Name: ____________________

Show your work. Each question is worth 10 points.

1. Find the points on the parametric curve

$$x = t^3 - 3t^2, \quad y = t^4 - 8t^2 + 16$$

where the tangent is horizontal or vertical.

Solution. We have $\frac{dx}{dt} = 3t^2 - 6t$ and $\frac{dy}{dt} = 4t^3 - 16t$. For a horizontal tangent, $4t^3 - 16t = 0$, $t = 0$ or ± 2. The points are $(0, 16)$, $(-20, 0)$ and $(-4, 0)$. For a horizontal tangent, $3t^2 - 6t = 0$, $t = 0$ or 2, giving two of the same points again. This is because there was a typo; x was supposed to be $t^3 - 3t$. With the curve as given, what you can do is simplify $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{4t^3 - 16t}{3t^2 - 6t} = \frac{4t(t-2)(t+2)}{3t(t-2)} = \frac{4}{3}(t+2),$$

so in fact there's a horizontal tangent where $t = -2$ (the point $(-20, 0)$), and no vertical tangent. □

2. Find the length of the parametric curve

$$x = t^3 - 1, \quad y = t^2 + 2, \quad 0 \le t \le 1.$$

Solution. We have $\frac{dx}{dt} = 3t^2$, $\frac{dy}{dt} = 2t$, $ds = \sqrt{9t^4 + 4t^2}\,dt = t\sqrt{9t^2 + 4}\,dt$. In the integral below, we substitute $u = 9t^2 + 4$, so $du = 18t\,dt$, $t = 0$ gives $u = 4$, and $t = 1$ gives $u = 13$. The length is

$$\begin{aligned} L &= \int_0^1 t\sqrt{9t^2 + 4}\,dt \\ &= \frac{1}{18}\int_4^{13} u^{1/2}\,du \\ &= \frac{1}{18}\left[\frac{2}{3}u^{3/2}\right]_4^{13} \\ &= \frac{1}{27}(13\sqrt{13} - 8). \end{aligned}$$

□

3. Find the area of one loop of the three-petaled rose $r = 4 \sin 3\theta$.

Solution. We have $r = 0$ where $3\theta = n\pi$, or $\theta = n\pi/3$, for n an integer. Two successive solutions are $\theta = 0$ and $\theta = \pi/3$, so one loop is given by $0 \le \theta \le \pi/3$ and its area is

$$\begin{aligned} A &= \frac{1}{2} \int_0^{\pi/3} 16 \sin^2 3\theta \, d\theta \\ &= \int_0^{\pi/3} 4(1 - \cos 6\theta) \, d\theta \\ &= 4\Big[\theta - \frac{1}{6} \sin 6\theta\Big]_0^{\pi/3} \\ &= \frac{4\pi}{3}. \end{aligned}$$

□

4. Find the center, vertices and foci of the ellipse

$$9x^2 + 25y^2 - 36x + 50y = 164.$$

Solution. We rewrite the equation.

$$\begin{aligned} 9(x^2 - 4x + 4 - 4) + 25(y^2 + 2y + 1 - 1) &= 164 \\ 9(x-2)^2 - 36 + 25(y+1)^2 - 25 &= 164 \\ 9(x-2)^2 + 25(y+1)^2 &= 225 \\ \frac{(x-2)^2}{25} + \frac{(y+1)^2}{9} &= 1. \end{aligned}$$

The center is at $(2, -1)$ and the focal axis is horizontal. We have $a = 5$, $b = 3$, and $c^2 = 25 - 9 = 16$, so $c = 4$. The vertices are $(2 \pm 5, -1) = (-3, -1)$ and $(7, -1)$ and $(2, -1 \pm 3) = (2, -4)$ and $(2, 2)$, and the foci are $(2 \pm 4, -1) = (-2, -1)$ and $(6, -1)$. □

5. For $\vec{u} = \langle 1, -2, 2\rangle$ and $\vec{v} = \langle 6, 3, 2\rangle$, find $3\vec{u} - 2\vec{v}$, the angle between $\vec{u}$ and $\vec{v}$, and $\vec{u} \times \vec{v}$.

Solution. We have

$$\begin{aligned} 3\vec{u} - 2\vec{v} &= \langle 3, -6, 6\rangle - \langle 12, 6, 4\rangle \\ &= \langle -9, -12, 2\rangle \\ \vec{u} \cdot \vec{v} &= 6 - 6 + 4 = 4 \\ |\vec{u}| &= \sqrt{1 + 4 + 4} = 3 \\ |\vec{v}| &= \sqrt{36 + 9 + 4} = 7 \end{aligned}$$

so the angle is $\cos^{-1} \frac{4}{21} \approx 1.379$. Also

$$\begin{aligned} \vec{u} \times \vec{v} &= \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & -2 & 2 \\ 6 & 3 & 2 \end{vmatrix} \\ &= \begin{vmatrix} -2 & 2 \\ 3 & 2 \end{vmatrix} \vec{\imath} - \begin{vmatrix} 1 & 2 \\ 6 & 2 \end{vmatrix} \vec{\jmath} + \begin{vmatrix} 1 & -2 \\ 6 & 3 \end{vmatrix} \vec{k} \\ &= -10\vec{\imath} + 10\vec{\jmath} + 15\vec{k}. \end{aligned}$$

□

6. For $\vec{u} = \langle 1, 2, 3\rangle$ and $\vec{v} = \langle 4, -1, 2\rangle$, find the decomposition $\vec{u} = \vec{u}_{\|} + \vec{u}_{\perp}$ with respect to $\vec{v}$.

Solution. We have

$$\vec{u}_{\|} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}\vec{v} = \frac{8}{21}\langle 4, -1, 2\rangle = \left\langle \frac{32}{21}, \frac{-8}{21}, \frac{16}{21} \right\rangle$$

and

$$\vec{u}_{\perp} = \langle 1, 2, 3\rangle - \left\langle \frac{32}{21}, \frac{-8}{21}, \frac{16}{21} \right\rangle = \left\langle \frac{-11}{21}, \frac{50}{21}, \frac{47}{21} \right\rangle.$$

□

7. Find the volume of the parallelepiped spanned by $\vec{u} = \langle -1, 0, 1\rangle$, $\vec{v} = \langle 1, 2, 3\rangle$ and $\vec{w} = \langle 2, 2, 1\rangle$.

Solution.

$$\begin{aligned}\det\begin{bmatrix}\vec{u} & \vec{v} & \vec{w}\end{bmatrix} &= \begin{vmatrix}-1 & 0 & 1\\ 1 & 2 & 3\\ 2 & 2 & 1\end{vmatrix}\\ &= -\begin{vmatrix}2 & 3\\ 2 & 1\end{vmatrix} - 0 + \begin{vmatrix}1 & 2\\ 2 & 2\end{vmatrix}\\ &= 4 - 2 = 2,\end{aligned}$$

so the volume is 2. □

8. Find the distance from the point $(6, -2, 1)$ to the plane $2x - 3y + z = 21$.

Solution. The distance is

$$\frac{|2(6) - 3(-2) + 1 - 21|}{\sqrt{4 + 9 + 1}} = \frac{2}{\sqrt{14}}.$$ □

9. State the type of the quadric surfaces below.

(a) $z = \frac{x^2}{5} + \frac{y^2}{2}$

(b) $-x^2 - \frac{y^2}{3} + \frac{z^2}{6} = 1$

Solution.

(a) Elliptic paraboloid.

(b) Hyperboloid of two sheets. □

10. Find a parametrization of the tangent line to the curve $\vec{r}(t) = \langle 2\cos t, \sin t, 3t\rangle$ at the point where $t = 0$.

Solution.

$$\begin{aligned}\vec{r}'(t) &= \langle -2\sin t, \cos t, 3\rangle\\ \vec{r}(0) &= \langle 2, 0, 0\rangle\\ \vec{r}'(0) &= \langle 0, 1, 3\rangle\end{aligned}$$

so the tangent line is

$$\vec{s}(t) = \langle 2, 0, 0\rangle + t\langle 0, 1, 3\rangle = \langle 2, t, 3t\rangle.$$ □