GLOBAL STABILIZATION FOR SYSTEMS EVOLVING ON MANIFOLDS

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STATEMENT of PROBLEM

Given System and Attractor:

$$\dot{x} = f(x,u), \quad x \in \mathcal{X}, \quad u \in \mathbf{U} \qquad (\Sigma)$$

Assumptions: f locally Lipschitz in x uniformly for u in compact sets, $\mathcal{X}$ a manifold, $\mathbf{U}$ a locally compact metric space.

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E.g.: Take $\mathcal{X} = \mathbb{R}^n$, $\mathcal{A} = \{0\}$, $\dot{x} = g_1(x)u_1 + \ldots + g_m(x)u_m$, $m > n$.
No continuous stabilizing $k(x)$ when $\text{rank}[g_1(0)\ g_2(0)\ \ldots\ g_m(0)] = m$.

OBSTACLES to CONTINUOUS STABILIZATION

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[Have Lyapunov function for closed loop system i.e. a Morse function with a unique possibly degenerate critical point. Use Milnor's Theorem.]

DEFINITIONS for SYSTEMS on $\mathcal{X} = \mathbb{R}^n$

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Trajectories of Σ: $x(\cdot, \xi, \mathbf{u})$ is unique maximally defined trajectory starting at ξ for the control $\mathbf{u} \in \mathcal{U}$. $|\cdot|_{\mathcal{A}}$ = distance to $\mathcal{A}$.
$\mathcal{U}$ = measurable essentially bounded functions $\mathbf{u} : [0, \infty) \to \mathbf{U}$.

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GAC of Σ: $\exists \beta \in \mathcal{KL}$ and $\sigma \in \mathcal{N}$ s.t. $\forall \xi \in \mathcal{X}$, there exists $\mathbf{u} \in \mathcal{U}$ for which $||\mathbf{u}|| \leq \sigma(|\xi|_{\mathcal{A}})$ and $|x(t, \xi, \mathbf{u})|_{\mathcal{A}} \leq \beta(|\xi|_{\mathcal{A}}, t)$ for all $t \geq 0$.
GAS of Σ: GAC with no controls.

SAMPLE SOLUTIONS on $\mathcal{X} = \mathbb{R}^n$

Partitions: $\pi : 0 = t_0 < t_1 < t_2 < \ldots < t_n \to +\infty$. (Sample times.)
$\overline{\mathbf{d}}(\pi) = \sup_{i\geq 0}(t_{i+1} - t_i)$ is (upper) diameter of π. Par = all partitions.

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- Solving $\dot{x}(t) = f(x(t), k(x(t_i)))$ on $[t_i, t_{i+1}]$ (or as far as trajectory can be extended on that interval) with $x(0) = \xi$.
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Main Points:

- Introduced by Clarke-Ledyaev-EDS-Subbotin (CLSS), IEEE/TAC'97. [Aka sample-and-hold solutions with fast sampling.]
- Only sample in controller $k(x)$.
- Allows $k(x)$ to be discontinuous hence sample stabilization.

CLSS THEOREM on $\mathcal{X} = \mathbb{R}^n$

s-Stabilizing Σ to $\mathcal{A}$: This means $\forall (r, R)$ with $0 < r < R$, there exist

(i) $M = M(R) > 0$ with $M(R) \to 0$ as $R \to 0$

(ii) $\delta = \delta(r, R) > 0$, and $T = T(r, R) > 0$

such that $\forall \pi \in \mathrm{Par}$ with $\overline{\mathbf{d}}(\pi) < \delta$ and $\xi \in \mathcal{B}_R(\mathcal{A})$, the corresponding sample trajectory $x(t) := x_\pi(t, \xi, k)$ for Σ satisfies:

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Theorem (CLSS'97): If Σ is GAC to $\mathcal{A}$ on $\mathcal{X} = \mathbb{R}^n$, then Σ admits a feedback that s-stabilizes the system to $\mathcal{A}$.

MAIN CONTRIBUTION: CLSS THEOREM ON MANIFOLDS

Main Steps for Generalization to Manifolds:

S1 Embed manifold $\mathcal{X}$ as closed submanifold $g(\mathcal{X}) \subseteq \mathbb{R}^k$.

S2 Extend f to $\mathbb{R}^k$ so that $g(\mathcal{X})$ is invariant for extended system and $\mathbb{R}^k \setminus g(\mathcal{X})$ can be controlled to a tubular neighborhood of $\mathcal{A}$.

S3 Apply original CLSS Theorem to embedded system on $\mathbb{R}^k$ to get s-stabilizing feedback $k(x)$.

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Technical Tools:

T1 Whitey Embedding Theorem. $k = 2n + 1$.

T2 Tubular neighborhood theorem.

ILLUSTRATION with $\mathcal{A} = \{\pm \mathrm{NP}\} \subseteq \mathbf{S}^2$

Dynamics:

$$\dot{x} = A_1(x)u_1 + A_2(x)u_2 \in T_x(\mathcal{X}), \quad u \in \mathbb{R}^2, \quad \mathcal{X} = S^2 \qquad (\Sigma')$$

where $A_1(x) = M_1(x)B_1(x)$, $A_2(x) = B_2(x)$, $B_1(x) = q - (x \cdot q)x$, $B_2(x) = x \times q$, $q = \mathrm{NP}$, and M_1 introduces a set of zeros in a geodesic rectangle covering part of the equator in $\{x \in S^2 : x_1 > 0, x_2 > 0\}$.

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Main Points: $\{B_1, B_2\}$ = basis for $T_x(\mathcal{X}) = \{x\}^\top$ outside $\mathcal{A}$. M_1 limits motions. We assume M is valued in $[0, 1]$. Discontinuous stabilizer:

$$u = K(x) = \begin{cases} \langle 1, 0 \rangle, & \text{if } \; x_3 \geq 0 \;\text{ and }\; M_1(x) = 1 \\ -\langle 1, 0 \rangle, & \text{if } \; x_3 < 0 \;\text{ and }\; M_1(x) = 1 \\ \langle 0, 1 \rangle, & \text{if } \; M_1(x) < 1 \end{cases}$$

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A corresponding explicit Lyapunov function can also be constructed.

LYAPUNOV FUNCTION for SPHERE EXAMPLE

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Main Points:

- V_r part of V puts high cost on being near $\pm r$.
- V continuous, only zero on $\mathcal{A}$, nonsmooth
- Control-Lyapunov integral function: There exist $N > 0$ and $\alpha_3 \in \mathcal{K}$ satisfying: $\forall \xi \in \mathcal{X}$, $\exists u \in \mathcal{U}_N$ s.t. $x(t) := x(t, \xi, u)$ satisfies

$$V(x(t)) - V(\xi) \leq - \int_0^t \alpha_3(|x(s)|_{\mathcal{A}}) ds \ \ \forall t \geq 0.$$

- $\mathcal{U}_N$ - $\{u \in \mathcal{U} : ||u|| \leq N\}$.

EXTENSIONS/CONCLUSIONS

- Can also sample input-to-state stabilize control affine systems with observation errors on manifolds a la [M-Rifford-EDS, SICON'04].
  - Use semiconcave CLF, with Riemannian metric to quantify observation error.
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- Can make fully nonlinear systems s-ISS in weak sense of EDS IEEE/TAC'90 "Further Facts.." paper on Riemannian manifolds.
- Alternative Approach
  - No embedding. Intrinsic.
  - Build CLFs and feedbacks on manifolds as we did for S^2 example.
  - Ongoing joint research by M with Ludovic Rifford.

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