Math 1551 (McGehee) Exercise List 1 September 2, 2002

E1. Prove by induction that for all natural numbers n,

$$1^2 + 2^2 + \cdots + n^2 \text{ (which means } \sum_{k=1}^{n} k^2) = \frac{n(n+1)(2n+1)}{6}.$$

E2. Prove by induction that for every positive integer n, the sum of all the odd integers from $2n+1$ up to and including $4n-1$ equals $3n^2$.

E3. For each positive integer n, let P_n denote the assertion that $n^2 + 5n + 1$ is an even integer.

a. Show that P_{n+1} is true whenever P_n is true.

b. For which values of n is P_n actually true? What is the point of this exercise?

E4. (Double credit) Prove by induction that for all natural numbers n,

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2,$$

that is,

$$\sum_{k=1}^{n} k^3 = (\sum_{k=1}^{n} k)^2.$$

E5. (Triple credit) Prove by induction that whenever $a_1, a_2, \cdots, a_n$ are positive real numbers, then $G \leq A$ where

$$G := (a_1 a_2 \cdots a_n)^{1/n} \quad \text{and} \quad A := \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

That is, the geometric mean of the numbers is less than or equal to their arithmetic mean. Another way to state this result is as follows: Let $a_1, a_2, \cdots$ be a sequence of positive real numbers. Then for each n, the statement

$$\mathbf{P_n}: \qquad G_n \leq A_n$$

is true; where G_n and A_n are the geometric and arithmetic means, respectively, of the first n numbers in the sequence. You may need some hints. Here are two hints that you may be able to use to put together a proof, albeit one that doesn't look like the usual induction proof. (1) You may find that you can prove that for each integer k,

$$\mathbf{P_{2^k}} \Rightarrow \mathbf{P_{2^{k+1}}}.$$

That is, $\mathbf{P_1}$ implies $\mathbf{P_2}$ implies $\mathbf{P_4}$ implies $\mathbf{P_8}$ and so forth. (2) You may find that you can prove that for each integer $n \geq 2$, $\mathbf{P_n} \Rightarrow \mathbf{P_{n-1}}$.

E6. (Double credit) For each positive integer n, let $f(n)$ be the maximum number of regions into which the plane can be divided by n lines. Discover a formula for $f(n)$ and prove by induction that it's the right formula.

E7. (Double credit) **(The Towers of Hanoi)** There are three poles. There are n disks of all different diameters, each with a hole at its center, so that it can be put on any of the poles. Begin with all n disks stacked on one of the poles, in order by size, largest disk at the bottom of the stack. For each positive integer n, let $f(n)$ be the minimum number of moves required to relocate the n disks all to a different pole. Discover a formula for $f(n)$ and prove by induction that it's the right formula. Here's what is meant by a move. A move picks up one disk from one pole and puts it down on another pole, so that if there are already one or more disks on the second pole, the disk that you relocate to that pole is the smallest one on the stack. (So aftr each move, there may be a stack of disks on each of one, two, or three of the poles; and the disks on each pole the are stacked in order by size, largest disk at the bottom of the stack.)

E8. Use the Rational Zeros Theorem to find all the rational zeros of the polynomial p given by $p(x) = 3x^3 - 2x^2 - 9x + 6$. State the Theorem, explain how you use it, and show all your steps.

E9. Using the Rational Zeros Theorem, prove that $(13)^{1/4}$ is not a rational number.

E10. Using the Rational Zeros Theorem, prove that $(2 + \sqrt{2})^{1/2}$ is not a rational number.

E11. Using the Rational Zeros Theorem, prove that $(3 + \sqrt{2})^{2/3}$ is not a rational number.

E12. Write up in detail a proof of this **Theorem:** Let $f : [a, b] \to \mathbf{R}$ be a continuous function, such that $f(a) > q > f(b)$. Then there exists a real number c such that $a < c < b$ and $f(c) = q$. **Note:** This is a version of the Intermediate Value Theorem, and the problem is essentially to use the Least Upper Bound Axiom to prove the existence of c. Notice how this version differs from the one done in class, in that $f(a)$ is greater than $f(b)$ and q is an arbitrary number between those two values.

E13. Let $P(x, 1/x)$ be a point traveling along the graph of the equation $y = 1/x$ $(x > 0)$ in such a way that its horizontal coordinate is increasing at a constant rate of 2 units per second. Let Q be the point of intersection of (1) the normal line to the graph at P and (2) the y-axis. How fast is Q moving down the y-axis at the instant of time when P is at the point $(2, 1/2)$? Present your procedure fully and clearly. (When writing up your solution, think about how you would present it to the class at the blackboard.)

E14. Let

$$f(x) = \begin{cases} x + x^2 \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

a. Sketch the graph on the interval $-1 < x < 1$ as best you can.
b. Find $f'(x)$. For $x = 0$, you'll need to use the definition (set up the difference quotient and identify the limit).
c. Does $\lim_{x \to 0} f'(x)$ exist? If so, what is it?