Math 2090 (McGehee) Final Exam December 5, 2000

1. Solve this heat flow problem by identifying the function $u(x,t)$:

   (1) $\dfrac{\partial u}{\partial t}(x,t) = 3\dfrac{\partial^2 u}{\partial^2 x}(x,t), \quad (0 < x < \pi,\ t > 0),$

   (2) $u(0,t) = u(\pi,t) = 0 \quad (t > 0),$

   (3) $u(x,0) = \sin x - 6\sin 4x, \quad (0 < x < \pi).$

2. Let f be the 2π-periodic function given by

$$f(x) = \begin{cases} -1 & \text{if } -\pi < x < 0; \\ 1 & \text{if } 0 < x < \pi. \end{cases}$$

   Then f has a Fourier series representation:

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

   Find the coefficients a_n and b_n in terms of n. Show your procedure.

3. Find the inverse Laplace transform of this function: $F(s) = \dfrac{e^{-s}}{s^2 - 1}$.

4. Solve this initial value problem by the method of Laplace transforms. Show your procedure.

$$y'' + y' + \frac{5}{4}y = \begin{cases} 1 & \text{for } 0 \le t < \pi, \\ 0 & \text{for } \pi \le t. \end{cases} \qquad y(0) = 0, y'(0) = 0.$$

5. Given: The eigenvalues of the matrix

$$A = \begin{pmatrix} 5 & 12 & -6 \\ -3 & -10 & 6 \\ -3 & -12 & 8 \end{pmatrix}$$

   are -1 and 2.

   a. Find the eigenspace of each eigenvalue.

   b. Find a matrix S such that $S^{-1}AS$ is a diagonal matrix.

   c. Find the determinant of A.

   d. Find the trace of A.

6. Let A be the matrix in problem 5.

   a. Find the general solution of $\mathbf{x}' = A\mathbf{x}$.

   b. Find a fundamental matrix for that problem.

7. Find an example of one 4×4 matrix A, which of course defines a linear map from $\mathbf{R}^4$ to $\mathbf{R}^4$, such that these two conditions are both satisfied:

(1) The vectors $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ are in the range of the map; and

(2) The vectors $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ are in the kernel of the map.

8. Using any method you like, find the unique solution of this initial value problem, and sketch its graph:

$$y'' + 3y' + 2y = 0, \quad y(0) = 1, \quad y'(0) = -3.$$

This problem describes a mass-spring system. Is the system underdamped or overdamped? Sketch the graph of the motion. Does $y(t) = 0$ for some $t > 0$? If so, what is the value of t?