Mathematics 2090 (McGehee) Test 2 September 28, 2000

1. Find an example of one 4×4 matrix A, which of course defines a linear map from $\mathbf{R}^4$ to $\mathbf{R}^4$, such that these two conditions are both satisfied:

   (1) The vectors $\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ are in the range of the map; and

   (2) The vectors $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ are in the kernel of the map.

2. Consider these vectors in $\mathbf{R}^3$:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

   They form a set that is independent but not orthogonal. If the Gram-Schmidt process is carried out, the result will be an orthogonal set of 3 vectors; what are they?

3. Consider the following five statements about a finite-dimensional vector space. Just write "True" or "False" on the left. You must give five correct responses to get an A on this problem; four will suffice for a C, three for a D.

   a. If $\{\mathbf{v_1}, \cdots, \mathbf{v_n}\}$ is an independent set, and if $\sum_{k=1}^{n} c_k \mathbf{v_k} = \mathbf{0}$, then each c_k equals 0.
   b. Every spanning set has a subset which is a basis.
   c. Every subset of an independent set is independent.
   d. Any two bases have the same number of elements.
   e. Any two spanning sets have the same number of elements.

4. Let $f_1(x) = \sin x$, $f_2(x) = \cos x$, and $f_3(x) = 1$. Compute the Wronskian $W[f_1, f_2, f_3](x)$. One can determine whether functions are independent by looking at their Wronskian. Are these three functions independent?

5. Find the unique solution of this initial value problem, and sketch its graph:

$$y'' + 3y' + 2y = 0, \quad y(0) = 1, \quad y'(0) = -3.$$

   This problem describes a mass-spring system. Is the system underdamped or overdamped? Sketch the graph of the motion. Does $y(t) = 0$ for some $t > 0$? If so, what is the value of t?

Partial Key

1. Two of the possible solutions are

$$A = \begin{pmatrix} -1 & 2 & 1 & 0 \\ -1 & 1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix} \text{ and } A = \begin{pmatrix} -2 & 1 & 2 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

2. The result is
$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$
3. True, True, True, True, False
4. The Wronskian $W[f_1, f_2, f_3](x)$ is, for each x, the determinant of this matrix:
$$\begin{pmatrix} \sin x & \cos x & 1 \\ \cos x & -\sin x & 0 \\ -\sin x & -\cos x & 0 \end{pmatrix}.$$
The value of the determinant is identically equal to -1. Since it is never equal to 0, the three functions must be independent.