Math 2090 (McGehee) **Exercises, List 1**

A. A function $y = g(x)$ is described by saying: At every point where the graph of g intersects a curve given by $y = x^2 + k$ (where k is a constant), the graph and the curve are normal to each other. Write a differential equation of the form $dy/dx = f(x, y)$ having the function g as a solution. Explain your reasoning, show your procedure. You're expected to write up a thorough, well-explained solution. Write it at the level of students in the class. It is advisable to draw a picture to show what's going on.

A. **Solution:** The answer is $y' = -\dfrac{1}{2x}$. You should provide an accurate sketch of the family of curves $y = x^2 + k$, and you should indicate how you got the answer, the pertinent fact being that if two lines are perpendicular (normal), then the product of their slopes is -1. Thus if the tangent to a curve has slope $2x$, then the normal has slope $-1/(2x)$. If you go beyond the instructions and find the solution to the DE that you have found, then that work and your sketches of the solution should be accurate. The solution can be found by the method of separation of variables, and it is $y = -\frac{1}{2}\ln x + c$.

B. If a woman can jump vertically to a height of 2.25 feet on the earth, how high can she jump on the moon, where the surface gravitational acceleration is approximatley 5.3 feet per second squared? Write up your solution with a thorough explanation of your procedure. You are entitled to assume that the jumper will be capable of the same *initial velocity* on the moon as on the earth. If you wish to make a different assumption, feel free, but explain your reasoning.

B. **Solution:** The equations for position (distance above the ground) $y(t)$, velocity $y'(t)$, and acceleration $y''(t)$ are as follows:

$$y''(t) = -g, \qquad y'(t) = -gt + v_0, \qquad y(t) = -\frac{g}{2}t^2 + v_0 t.$$

We take the initial position, ground level, to be $y(0) = 0$. The initial velocity v_0, an instantaneous velocity depending on the physical ability of the jumper, is presumed to be the same on earth or moon. The maximum height occurs at the time t^*, and that's the time when velocity is zero,

$$y'(t^*) \equiv -gt^* + v_0 = 0.$$

So

$$t^* = \frac{v_0}{g}.$$

We are given that on earth, the maximum height is 2.25 feet. So

$$y\left(\frac{v_0}{g}\right) = 2.25 \equiv \frac{9}{4}.$$

Since $g = 32$ feet per second per second on earth, we may use the equation for $y(t)$ and solve for v_0:

$$\frac{9}{4} = y\left(\frac{v_0}{32}\right) = -16\left(\frac{v_0}{32}\right)^2 + v_0\left(\frac{v_0}{32}\right) \equiv \frac{v_0^2}{64}.$$

We obtain that

$$v_0 = 12 \text{ feet per second.}$$

Now we go to the moon, where $g = 5.3$ feet per second per second and thus

$$y'(t) = -5.3t + 12, \qquad y(t) = -2.65t^2 + 12t;$$

so $y'(t) = 0$ when $t = 12/5.3 \approx 2.26$. That's the time of maximum height, and that height is

$$\approx y(2.26) \approx 13.58 \text{ feet.}$$

C. The half-life of radioactive cobalt is 5.27 years. Suppose that a nuclear accident has left the level of cobalt radiation in a certain region at 100 times the level acceptable or human habitatio. How long will it be before the region is again habitable? Ignore the probable presence of other radioactive isotopes.

C. **Solution:** The amount $x(t)$ of a radioactive substance at time t will be given by $x(t) = x(0)e^{kt}$ for some constant k. If we know the half-life, then we can determine k. In this case, the half-life is 5.27 years, so k satisfies $e^{5.27k} = 0.5$. Thus

$$k = \frac{\ln .5}{5.27} \approx -0.13.$$

If $k = -0.13$, then

$$x(t) = x(0)e^{-0.13t},$$

and in this problem we want to know how many years it takes for the amount of the substance, in this case radioactive cobalt, to be reduced by a factor of .01. That is, we want the value of t such that

$$e^{-0.13t} = 0.01.$$

The answer is

$$t = \frac{\ln .01}{-0.13} \approx 35 \text{ years.}$$

D. Consider the initial value problem

$$\frac{dx}{dt} = ax - bx^2, \quad \text{where} \quad a = 0.01 \text{ and } b = 0.0001; \qquad x(0) = 50.$$

Show in detail how to use the separation of variables method to find the solution. Make a reasonably accurate sketch of the graph of the solution $x(t)$. Identify

$$\lim_{t\to\infty} x(t) \qquad \text{and} \qquad \lim_{t\to-\infty} x(t).$$

D. **Solution:** The differential equation, with variables separated, is

$$\frac{dx}{x(100-x)} = \frac{dt}{10,000}.$$

Partial fractions decomposition of the integrand on the left-hand side proceeds as follows:

$$\frac{1}{x(100-x)} = \frac{A}{x} + \frac{B}{100-x} \text{ (for some } A \text{ and } B) = \frac{100A - Ax + Bx}{x(100-x)},$$

from which we conclude that $A = 1/100 = B$:

$$\frac{1}{x(100-x)} = \frac{1}{100}\left(\frac{1}{x} + \frac{1}{100-x}\right).$$

So, making use of the initial value, we see that the integral becomes

$$\int_{50}^{X} \left(\frac{1}{x} + \frac{1}{100-x}\right) dx = \frac{1}{100}\int_0^T dt.$$

Thus

$$\ln \frac{x}{100-x}\bigg|_{50}^{X} = \frac{T}{100} \quad \text{or} \quad \ln \frac{X}{100-X} = \frac{T}{100}.$$

Solve for X in terms of T and you get

$$X = \frac{100e^{T/100}}{1+e^{T/100}},$$

from which one can see that the two limits are 100 and 0 respectively. The picture of the direction field also makes the limits clear. Note that a sketch of the curve is also required.

E. Consider the initial value problem

$$\frac{dy}{dx} = -\frac{1}{2x}, \qquad y(1) = 0.$$

Show in detail how to use the separation of variables method to find the solution. Make a reasonably accurate sketch of the graph of the solution $y(x)$. Find the values of

$$\lim_{x\to\infty} y(x) \qquad \text{and} \qquad \lim_{x\to 0^+} y(x).$$

E. **Solution:** Separation of variables leads to

$$dy = -\frac{1}{2}\frac{dx}{x}, \quad \text{hence} \quad \int_0^Y dy = -\frac{1}{2}\int_1^X \frac{dx}{x}.$$

Carry out the integration, and you get

$$Y = -\frac{1}{2}\ln X.$$

Th two limits are $-\infty$ and ∞, respectively.

F. Consider the first-order linear DE

$$xy' + 2y = 4x^2.$$

Show in detail how to find the general solution by the integrating-factor method. Also, find the solution such that $y(1) = 2$ and make a sketch of its graph.

F. **Solution:** We should start with the DE in the form

$$y' + \frac{2}{x}y = 4x.$$

The integrating factor will be

$$\rho(x) = e^{\int \frac{2dx}{x}} = e^{2\ln x} = x^2.$$

Multiply both sides of the DE by x^2, obtaining

$$x^2y' + 2xy = 4x^3 \quad \text{or} \quad (x^2y)' = 4x^3,$$

so that $x^2y = x^4 + C$ and the general solution is given by

$$y = x^2 + \frac{C}{x^2}.$$

Using $y(1) = 2$ we may solve for C, obtaining $C = 1$, so that the solution to the IVP is

$$y = x^2 + \frac{C}{x^2}.$$

G. Suppose that a water tank has a hole of area a at its bottom, through which water flows out. Denote by $y(t)$ the depth, and by $V(t)$ the volume, of water in the tank at time t. Torricelli's Law states that under certain conditions, the velocity of water exiting through the whole is $v = \sqrt{2gy}$. Notice that that's the velocity a point-mass would acquire in falling freely from the surface of the water to the hole. Write up a derivation and explanation of Torricelli's Law. You should write for a student on your level, and explain everything clearly. Your work will be graded in part on the skill of your exposition. The Law is usually derived not directly from conservation of energy, but by way of Bernoulli's Law. (There's a derivation in the LSU physics text, *Fundamentals of Physics, Extended*, fifth edition, by Halliday, Resnick, and Walker, published by John Wiley & Sons, Inc.; but in that book, it is not called Torricelli's Law.)

H. (The *clepsydra*, or water clock.) A 12-hour water clock is designed so that the containing surface is the surface of revolution obtained by revolving the graph of $y = f(x)$, $0 \le x \le 1$ about the $y-$axis. It is given that the units are feet, f is strictly increasing on $[0, 1]$, and $f(1) = 4$; thus the top of the tank is a circular opening of radius 1, and the tank is 4 feet tall. But then a very small circular hole, radius r, is cut in the bottom of the tank, by means of a horizontal cut. What should f be, and what should r be, in order that the water level will fall at the constant rate of 4 inches per hour? (Since the hole is said to be very small, assume that the depth of the water is $f(x)$ when the surface of the water comes up to the point $(x, f(x))$, even though a bit of the tank has been cut away.)

H. **Solution:** Let $y = y(t)$ be the depth of the water at time t. Since it's a 12-hour clock in which the depth is initially 4 feet, we know that $y(0) = 4$ hours and $y(12) = 0$. Let a be the area of the circular hole at the bottom; we want to find r such that $a = \pi r^2$. We are given that $y'(t) = -4$ in/hr, which is the same as $-(1/10800)$ ft/sec. That's the constant rate at which the water level falls. The reservoir is the surface of revolution obtained when a curve $y = f(x)$ revolves about the y-axis. We want to identify f. Torricelli's Law implies that if $A(y)$ is the cross-sectional area of the reservoir at depth y, then

$$A(y)y'(t) = -a\sqrt{2gy} \equiv -\pi r^2\sqrt{64y}.$$

$$\pi x^2(-1/10800) = -8\pi r^2\sqrt{y}.$$

Therefore the relation between y and x has the form $y = cx^4$ for some constant c. Since $y(1) = 4$, c must be 4, and we may write $4x^4$ for y. It follows that

$$-\frac{\pi x^2}{10800} = -8\pi r^2 2x^2$$

or

$$r = \sqrt{\frac{1}{8 \times 10800 \times 2}} \approx 0.00238 \text{ feet } \approx 0.029 \text{ inches }.$$

I. Consider the differential equation

$$x'' + x' + \frac{5}{4}x = 0.$$

a. Find the general real-valued solution.
b. Find the unique solution such that $x(0) = 0$ and $x'(0) = 1$.
c. Make an accurate sketch of that solution, over the interval $0 \le t \le \pi$. You may use a computer to provide this sketch if you like, but a careful pen-or-pencil sketch will be OK.
d. Show how to use calculus methods to find the value of t at which the solution $x(t)$ attains its maximum value, and find that maximum value. Your sketch should show this information correctly.

I. **Solution, in part:** The general solution is given by

$$x(t) = c_1e^{-t/2}\cos t + c_2e^{-t/2}\sin t.$$

The unique solution of the IVP is $x(t) = e^{-t/2}\sin t$. As we can see from the graph, the maximum occurs at the smallest value of t at which the derivative

$$x'(t) = -\frac{1}{2}e^{-t/2}\sin t + e^{-t/2}\cos t \equiv -\frac{1}{2}e^{-t/2}(\sin t - 2\cos t)$$

equals 0, which is where $\sin t = 2\cos t$ or $\tan t = 2$, approximately 1.1 radians. The maximum $x(\arctan 2)$ is approximately .51.

J. Find the unique solution of this initial value problem. Show your procedure clearly and completely.

$$y'y'' = \frac{1}{2}, \quad y(0) = 1, \quad y'(0) = 0.$$

Hint: If you let $v = y'$, you obtain a first-order equation in v,which can be solved by separation of variables.

J. **Solution:** If $v = y'$, the equation becomes

$$v\frac{dv}{dx} = \frac{1}{2}, \quad \text{so} \quad v\,dv = \frac{1}{2}dx.$$

Using the initial value data and integrating, we obtain

$$\int_0^V v\,dv = \frac{1}{2}\int_0^X dx, \quad \text{hence} \quad \frac{V^2}{2} = \frac{1}{2}X.$$

Reverting to lower case letters to express the conclusion, we have $v = x^{1/2}$, which is to say $y' = x^{1/2}$, and hence $y = \frac{2}{3}x^{3/2} + C$. Using the rest of the initial value information, we see that $y(0) = C = 1$. Thus:

$$y = \frac{2}{3}x^{2/3} + 1.$$

K. A cube of side length L and uniform density ρ_0 floats when half submerged in a fluid of density ρ. At time $t = 0$ the mass is pushed down a distance x_0 (smaller than $L/2$) and released. Determine the initial value problem (DE together with prescribed initial values) that governs the resulting motion. Assume that all motion takes place, and all forces act, along a vertical x-axis, with the downward direction positive. Hint: Remember Archimedes's Principle.

K. **Solution:** Force due to gravity (the weight of the cube): $l^3\rho_0 g$. Force due to the fluid (upward, equal to the weight of the fluid displaced, by Archimedes's Principle): $-L(\frac{L}{2} + x)\rho g$. The zero on the x-axis is where the cube floats at equilibrium. By Newton's Law "$F = ma$," then, we have

$$L^3\rho_0 g - L^2\,(L/2 + x)\,\rho g = L^3\rho x''.$$

Since $x = 0$ gives equilibrium, we know $L^3\rho_0 g - L^2(\frac{L}{2})\rho g = 0$, so $\rho_0 = \rho/2$, and the equation becomes

$$-2L^2\rho g x = L^3\rho x''.$$

So the problem may be stated as follows:

$$x'' + \frac{2g}{L}x = 0, \quad x(0) = x_0.$$

L. On page 139 of the text, the equation of motion which is the solution to a one-mass, one-spring vibrating system is re-written in the phase-angle form, in which the amplitude and phase angle appear prominently. Consider the initial value problem

$$3x'' + 24x' + 192x = 0, \quad x(0) = 1, \quad x'(0) = 0,$$

which we may intepret as describing a vibrating system. Find the unique solution, showing your procedure clearly. Also, show how to convert the solution into phase-angle form. Identify the time-varying amplitude, the frequency, and the phase angle.

L. **Solution:** The DE may be written $x'' + 8x' + 64x = 0$, and the zeros of the characteristic polynomial $r^2 + 8r + 64$ are $-4 \pm i4\sqrt{3}$. So the general solution is

$$x(t) = e^{-4t}(A\cos 4\sqrt{3}t + B\sin 4\sqrt{3}t).$$

When we apply the initial-value information, we find that $A = 1$ and $B = 1/\sqrt{3}$. So

$$\begin{aligned} x(t) =& e^{-4t}\left(\cos 4\sqrt{3}t + \frac{1}{\sqrt{3}}\sin 4\sqrt{3}t\right) \\ =& e^{-4t}\frac{2}{\sqrt{3}}\left(\frac{\sqrt{3}}{2}\cos 4\sqrt{3}t + \frac{1}{2}\sin 4\sqrt{3}t\right). \end{aligned}$$

Noting that $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and $\sin\frac{\pi}{6} = \frac{1}{2}$, we see that we can write the solution as follows:

$$x(t) = \frac{2}{\sqrt{3}}e^{-4t}\cos\left(4\sqrt{3}t - \frac{\pi}{6}\right).$$

That's the phase-angle form. So the time-varying amplitude is $\frac{2}{\sqrt{3}}$ feet; the frequency is $4\sqrt{3}$ radians per second; and the phase angle is $\pi/6$ radians.

M. Make up a simple example of a second-order linear constant-coefficient homogeneous differential equation that describes a vibrating system with critical damping. Find the general solution of the equation you select. Then make up initial values so that the resulting unique solution $x(t)$ has the property that $x(t) = 0$ for one and only one time $t > 0$. Find that value of t.

M. **Solution:** One possibility is

$$x'' + 2x' + x = 0, \quad x(0) = 1, \quad x'(0) = -1.$$

The solution is then $x(t) = e^{-t}(1-t)$, which equals 0 when $t = 1$.

N. Find the general solution of the differential equation

$$x'''' + 28x'' + 96x = 0.$$

N. **Solution, in part:** The characteristic polynomial factors as follows:

$$r^4 + 28r^2 + 96 = (r^2 + 24)(r^2 + 4) = (r + 2\sqrt{6}i)(r - 2\sqrt{6}i)(r + 2i)(r - 2i).$$

Therefore the general solution is

$$x(t) = C_1 \cos 2t + C_2 \sin 2t + C_3 \cos 2\sqrt{6}t + C_4 \sin 2\sqrt{6}t.$$

O. Consider the differential equation

$$x'''' + 7x'' + 6x = 0.$$

a. Find the general solution.

b. Find the unique solution determined by the initial values

$$x(0) = 1, \quad x'(0) = 0, \quad x''(0) = -6, \quad x'''(0) = 0.$$

Show a complete procedure for obtaining the solutions to a and b by hand. Of course, you are free to check your work by using DSolve.

O. **Solution:** The characteristic polynomial is a quadratic in r^2 which factors easily into two quadratics in r which also factor easily:

$$r^4 + 7r^2 + 6 = (r^2)^2 + 7(r^2) + 6 = (r^2 + 6)(r^2 + 1) = (r + i\sqrt{6})(r - i\sqrt{6})(r + i)(r - i).$$

From the factorization we can conclude that the general real-valued solution may be written

$$x(t) = C_1 \cos \sqrt{6}t + C_2 \sin \sqrt{6}t + C_3 \cos t + C_4 \sin t.$$

Now let's take the first three derivatives of x:

$$x'(t) = -\sqrt{6}C_1 \sin \sqrt{6}t + \sqrt{6}C_2 \cos \sqrt{6}t - C_3 \sin t + C_4 \cos t,$$

$$x''(t) = -6C_1 \cos \sqrt{6}t - 6C_2 \sin \sqrt{6}t - C_3 \cos t - C_4 \sin t,$$

$$x'''(t) = 6\sqrt{6}C_1 \sin \sqrt{6}t - 6\sqrt{6}C_2 \cos \sqrt{6}t + C_3 \sin t - C_4 \cos t.$$

From the given initial conditions we obtain four linear equations in the four unknown constants C_k:

$$\begin{aligned} C_1 \qquad\quad + C_3 \qquad\quad &= 1, \\ \sqrt{6}C_2 \qquad\quad + C_4 &= 0, \\ -6C_1 \qquad\quad - C_3 \qquad\quad &= -6, \text{ and} \\ -6\sqrt{6}C_2 \qquad\quad - C_4 &= 0. \end{aligned}$$

The solution to that system is $C_1 = 1,\ C_2 = C_3 = C_4 = 0$. Thus our answer is

$$x(t) = \cos\sqrt{6}t.$$

P. (Your objective is to prepare yourself a well-explained and well-written example of how to carry out the procedures. Of course, you are welcome to use DSolve to check your answers.) Consider the nonhomogeneous differential equation

$$y'' + 5y' + 6y = \cos 2t. \tag{1}$$

Notice that the characteristic polynomial is $r^2 + 5r + 6 \equiv (r+2)(r+3)$, so that the solution of the associated homogeneous problem is

$$y = C_1 e^{-2t} + C_2 e^{-3t}.$$

a. Show in detail how to find a particular solution of (1) by the method of judicious guessing (undetermined coefficients).
b. Find the general solution of (1). Remember that it should contain two undetermined constants.
c. Find the unique solution of (1) such that $y(0) = 2$ and $y'(0) = -5$.

P. **Solution:** The reasonable guess is that there is a solution of (1) of this form:

$$y = A\cos 2t + B\sin 2t \quad \text{for some constants } A \text{ and } B. \tag{2}$$

Then $y' = -2A\sin 2t + 2B\cos 2t$ and $y'' = -4A\cos 2t - 4B\sin 2t$. Now we'll try the y given by our guess in (2) in equation (1), to see if we can find A and B so that it will be a solution. Let's organize the result as a combination of the two independent functions $\cos 2t$ and $\sin 2t$:

$$y'' + 5y' + 6 = (2A + 10B)\cos 2t + (-10A + 2B)\sin 2t,$$

which according to (1) must equal $\cos 2t$. Thus we obtain two linear equations in the two unknowns A and B:

$$\begin{aligned} 2A + 10B &= 1, \\ -10A + 2B &= 0. \end{aligned}$$

Solving, we find that

$$A = \frac{1}{52} \quad \text{and} \quad B = \frac{5}{52}.$$

So we have found a particular solution of (1):

$$y_p(t) = \frac{1}{52}\cos 2t + \frac{5}{52}\sin 2t.$$

Thus the general solution of (1) is

$$y(t) = C_1 e^{-2t} + C_2 e^{-3t} + \frac{1}{52}\cos 2t + \frac{5}{52}\sin 2t.$$

Taking the derivative, we get

$$y'(t) = -2C_1 e^{-2t} - 3C_2 e^{-3t} - \frac{2}{52}\sin 2t + \frac{10}{52}\cos 2t.$$

The initial value information, $y(0) = 2$ and $y'(0) = -5$, gives us two linear equations in the two unknowns C_1 and C_2:

$$\begin{aligned} C_1 + \ \ C_2 + \frac{1}{52} &= 2 \text{ and} \\ -2C_1 - 3C_2 + \frac{10}{52} &= -\ 5. \end{aligned}$$

Adding twice the first equation to the second gives $-C_2 + \frac{12}{52} = -1$, so that $C_2 = \frac{64}{52}$. Using the first equation again, we get $C_1 = \frac{39}{52}$. So

$$y(t) = \frac{39}{52}e^{-2t} + \frac{64}{52}e^{-3t} + \frac{1}{52}\cos 2t + \frac{5}{52}\sin 2t.$$