Math 2090 (McGehee) **Exercises, List 2**

Q. In this Exercise, you are to show in detail how to obtain your results, by hand, and explain what you are doing. Of course, you're welcome to use a computer to check your work. Consider a two-mass, two-spring system with no damping. We've considered this type of system in class. If the spring constants are $k_1 = 3$ and $k_2 = 2$, and both masses are 1 unit, then we obtain a system of two DEs in the displacement functions u_1 and u_2:

$$\begin{aligned} u_1'' + 5u_1 &= 2u_2, \\ u_2'' + 2u_2 &= 2u_1. \end{aligned}$$

Using the method of elimination (such as will demonstrated in class with a similar problem), we obtain one fourth-order DE in u_1 only:

$$u_1'''' + 7u_1'' + 6u_1 = 0.$$

We find that the general solution for u_1 is

$$u_1(t) = C_1 \cos\sqrt{6}t + C_2 \sin\sqrt{6}t + C_3 \cos t + C_4 \sin t. \tag{1}$$

From that we can determine the equation for u_2, the displacement of the second mass:

$$u_2(t) = -\frac{1}{2}C_1 \cos\sqrt{6}t - \frac{1}{2}C_2 \sin\sqrt{6}t + 2C_3 \cos t + 2C_4 \sin t. \tag{2}$$

a. Find the initial values (that is, prescribed values for $u_1(0), u_1'(0), u_2(0), u_2'(0)$) that will result in the rather simple movement of the system, in which the two masses are always moving in opposite directions, given by these equations:

$$u_1(t) = -2\cos\sqrt{6}t, \qquad u_2(t) = \cos\sqrt{6}t. \tag{3}$$

b. Find the initial values that will result in the different, but still rather simple, movement–in which the two masses are always moving together–given by these equations:

$$u_1(t) = \cos t, \qquad u_2(t) = 2\cos t. \tag{4}$$

Q. **Solution, in Outline:** The more familiar kind of question would be: Given the initial values $u_1(0)$, $u_1'(0)$, $u_2(0)$, and $u_2'(0)$, determine the constants C_1, C_2, C_3, and C_4 that appear in the expression for the general solution, thereby obtaining the solution to the initial value problem. This Exercise is the reverse: Given the solution (first the one in (3), then the one in (4)), find the initial conditions that lead to that solution. Here's what you need to do. Obtain from (1) and (2) two more equations, giving $u_1'(t)$ and $u_2'(t)$. Then evaluate both sides of each of the four equations at $t = 0$, obtaining a system of four linear equations in the four unknown coefficients C_k:

$$\begin{aligned} C_1 \qquad\qquad + \quad C_3 \qquad\qquad &= u_1(0), \\ \sqrt{6}C_2 \qquad\qquad + C_4 &= u_1'(0), \\ -\frac{1}{2}C_1 \qquad\qquad + 2C_3 \qquad\qquad &= u_2(0), \\ -\sqrt{\frac{3}{2}}C_2 \qquad\qquad + 2C_4 &= u_2'(0). \end{aligned}$$

The question in part a is to determine the values for the four numbers on the right-hand sides of those four equations such that, when the system is solved for the constants C_k, the result will be

$$C_1 = -2, \quad C_2 = C_3 = C_4 = 0,$$

leading to the equations of motion given in (3). Looking at the four equations makes it clear that those values are $u_1(0) = -2$, $u_1'(0) = 0$, $u_2(0) = 1$, and $u_2'(0) = 0$. In physical terms: You push the first mass upward 2 units from equilibrium, and the second mass downward one unit, hold them still, then let go. Part b is quite similar. This time, in order to have the equations of motion (4), we want the solution to be $C_3 = 1$, $C_1 = C_3 = C_4 = 0$. And then the initial conditions must be $u_1(0) = 1$, $u_1'(0) = 0$, $u_2(0) = 2$, and $u_2'(0) = 0$. Stated physically: One pulls the first mass down one unit and the second mass down 2 units, holds them still, and then lets go.

R. Consider a vibrating system described by the initial value problem

$$u'' + \frac{1}{4}u' + 2u = 2\cos\omega t\,, \quad u(0) = 0, \quad u'(0) = 2.$$

1. Determine the steady-state part of the solution of this problem.
2. Find the amplitude A of the steady-state solution in terms of ω.
3. Plot A as a function of ω.
4. Find the maximum value of A and the frequency ω for which it occurs.

S. Consider the forced, undamped system described by the initial value problem

$$u'' + u = 3\cos\omega t\,, \quad u(0) = 0, \quad u'(0) = 0.$$

1. Find the solution $u(t)$ assuming $\omega \neq 1$.
2. Plot the solution $u(t)$ for each of the cases $\omega = 0.7$, $\omega = 0.8$, $\omega = 0.9$. Describe how the response $u(t)$ changes as ω takes on values closer and closer to 1.
3. Find the solution $u(t)$ for the case when $\omega = 1$ and plot the graph.

T. Consider the vibrating system described by the initial value problem

$$u'' + u = 3\cos\omega t\,, \quad u(0) = 1, \quad u'(0) = 1.$$

1. Find the solution $u(t)$ assuming $\omega \neq 1$.
2. Plot the solution $u(t)$ for each of the cases $\omega = 0.7$, $\omega = 0.8$, $\omega = 0.9$. Compare the results with those of Problem S. That is, describe the effect of having nonzero initial conditions.

U. Show in detail how to find the general solution of the DE

$$y'' + 4y = 3x^3.$$

First, find the general solution of the corresponding homogeneous solution. To find a particular solution, use the method of undetermined coefficients, noting that a reasonable guess is that there's a solution of the form

$$y = Ax^3 + Bx^2 + Cx + D.$$

That will lead to four linear equations in the four unknown coefficients A, B, C, D.

V. Give an example of a DE, with prescribed initial values, that governs the motion of a driven one-mass, one-spring vibrating system in which *beats* occur. Solve it and graph the solution.

W. Consider the DE

$$y'' + 3y' + 2y = 4e^x.$$

Show in detail how to find a particular solution (1) first using the method of undetermined coefficients, and (2) then using the method of variation of parameters.