Math 2090 (McGehee) Problem Set 5, Exercises Due Wednesday, October 20, 2004

A. Given: The eigenvalues of the matrix

$$A = \begin{pmatrix} 5 & 12 & -6 \\ -3 & -10 & 6 \\ -3 & -12 & 8 \end{pmatrix}$$

are -1 and 2.

a. Show how to find by hand the eigenspace of each eigenvalue. Of course, you may use a computer to verify your results.

b. Find a matrix S such that $S^{-1}AS$ is a diagonal matrix.

c. Find the determinant of A.

d. Find the trace of A.

A. **Solution:**

a. The eigenspace belonging to -1 is the null space of the matrix

$$\begin{pmatrix} 6 & 12 & -6 \\ -3 & -9 & 6 \\ -3 & -12 & 9 \end{pmatrix}.$$

Gaussian elimination leads to the reduced row-echelon matrix $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$. So the eigenspace of -1 is one-dimensional, and is spanned by $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$. The eigenspace belonging to 2 is the nullspace of the matrix $\begin{pmatrix} 3 & 12 & -6 \\ -3 & -12 & 6 \\ -3 & -12 & 6 \end{pmatrix}$; Gaussian elimination leads to the reduced row-echelon matrix $\begin{pmatrix} 1 & 4 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. So the eigenspace is two-dimensional and consists of all vectors $c_1 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}$.

b. The eigensystem information can be summarized in the following equation:

$$A.\underbrace{\begin{pmatrix} -1 & 2 & -4 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}}_{S} = \begin{pmatrix} 1 & 4 & -8 \\ -1 & 0 & 2 \\ -1 & 2 & 0 \end{pmatrix}, \text{ which equals } S.\underbrace{\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}}_{D}.$$

Thus $AS = SD$, and it follows that $A = SDS^{-1}$ and $D = S^{-1}AS$.

c. The determinant of A is the product of the eigenvalues, -4.

d. The trace of $A_{3\times 3}$ is the sum of the eigenvalues, 3.

B. Let A be the matrix in Exercise A and consider the problem $\mathbf{x}' = A\mathbf{x}$.

a. Find the general solution.

b. Find a fundamental matrix for the problem.

c. Find the transition matrix for the problem.

d. Find the unique solution such that $\mathbf{x}(0) = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$.

B. **Solution:** The general solution can be written using a fundamental matrix and a 3-vector of constants:

$$\mathbf{x}(t) = \underbrace{\begin{pmatrix} -e^{-t} & 2e^{2t} & -4e^{2t} \\ e^{-t} & 0 & e^{2t} \\ e^{-t} & e^{2t} & 0 \end{pmatrix}}_{\Phi(t)} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

Then

$$\Phi(0) = \begin{pmatrix} -1 & 2 & -4 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \Phi(0)^{-1} = \begin{pmatrix} 1 & 4 & -2 \\ -1 & -4 & 3 \\ -1 & -3 & 2 \end{pmatrix}.$$

The transition matrix at $t = 0$ is

$$e^{At} = \Phi(t)\Phi(0)^{-1} = \begin{pmatrix} -e^{-t} + 2e^{2t} & -4e^{-t} + 4e^{2t} & 2e^{-t} - 2e^{2t} \\ e^{-t} - e^{2t} & 4e^{-t} - 3e^{2t} & -2e^{-t} + 2e^{2t} \\ e^{-t} - e^{2t} & 4e^{-t} - 4e^{2t} & -2e^{-t} + 3e^{2t} \end{pmatrix}.$$

One may also compute it as $Se^{Dt}S^{-1}$. The transition matrix times $\begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$ gives the answer to the last part:

$$\mathbf{x}(t) = \begin{pmatrix} -7e^{-t} + 12e^{2t} \\ 7e^{-t} - 5e^{2t} \\ 7e^{-t} - 4e^{2t} \end{pmatrix}.$$

C. **(Double Credit)** Find the solution of the problem

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \\ 17 \end{pmatrix},$$

where

$$A = \begin{pmatrix} 3 & 0 & 1 \\ 9 & -1 & 2 \\ -9 & 4 & -1 \end{pmatrix}.$$

You may obtain the eigensystem by computer, if you wish.

C. **Solution** The eigensystem is as follows. The eigenvalues are 3, $-1-i$, and $-1+i$. The eigenspaces are generated respectively by the eigenvectors $\begin{pmatrix} 4 \\ 9 \\ 0 \end{pmatrix}$, $\begin{pmatrix} -4+i \\ -9-2i \\ 17 \end{pmatrix}$, and $\begin{pmatrix} -4-i \\ -9+2i \\ 17 \end{pmatrix}$. (Note: If your eigenvectors look different, they might still be correct. For example, the last eigenvector, multiplied by the constant $(-4+i)/17$ is $\begin{pmatrix} 1 \\ 2-i \\ -4+i \end{pmatrix}$.)

Thus the general complex-valued solution may be written

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 4 \\ 9 \\ 0 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} -4+i \\ -9-2i \\ 17 \end{pmatrix} e^{(-1-i)t} + c_3 \begin{pmatrix} -4-i \\ -9+2i \\ 17 \end{pmatrix} e^{(-1+i)t}. \qquad (1)$$

Next we will write the general real-valued solution, in convenient form, and then we will use the initial values and obtain the solution. (Note that if we preferred, we could proceed to use the initial values right away and skip getting the representation of the real-valued solution.) To get the optimal form of the general real-valued solution, let's first take the second part of (1) and re-write it:

$$\begin{aligned} &\begin{pmatrix} -4+i \\ -9-2i \\ 17 \end{pmatrix} (e^{-t}\cos t - ie^{-t}\sin t) \\ &\qquad = \begin{pmatrix} -4e^{-t}\cos t + e^{-t}\sin t \\ -9e^{-t}\cos t - 2e^{-t}\sin t \\ 17e^{-t}\cos t \end{pmatrix} + i \begin{pmatrix} 4e^{-t}\sin t + e^{-t}\cos t \\ e - t\cos t + 9e^{-t}\sin t \\ -17e^{-t}\sin t \end{pmatrix}. \end{aligned}$$

If we process the third summand in (1) the same way, we get the same thing complex-conjugated. So we may write the general real-valued solution as follows:

$$\mathbf{x}(t) = C_1 \begin{pmatrix} 4e^{3t} \\ 9e^{3t} \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} -4e^{-t}\cos t + e^{-t}\sin t \\ -9e^{-t}\cos t - 2e^{-t}\sin t \\ 17e^{-t}\cos t \end{pmatrix} + C_3 \begin{pmatrix} 4e^{-t}\sin t + e^{-t}\cos t \\ e^{-t}\cos t + 9e^{-t}\sin t \\ -17e^{-t}\sin t \end{pmatrix}.$$

Now, applying the initial-value information, we find three equations in the three unknowns C_k:

$$\begin{aligned} 4C_1 - 4C_2 + C_3 &= 0, \\ 9C_1 - 9C_2 + C_3 &= 0, \\ 17C_2 \qquad &= 17. \end{aligned}$$

The constants are $C_1 = C_2 = 1,\ C_3 = 0$, so the solution to the IVP is

$$\begin{aligned} x_1(t) =& 4e^{3t} - 4e^{-t}\cos t + e^{-t}\sin t,\\ x_2(t) =& 9e^{3t} - 9e^{-t}\cos t - 2e^{-t}\sin t,\\ x_3(t) =& 17e^{-t}\cos t. \end{aligned}$$

Suggestion for further study: Find e^{At}.

D. Show in detail how to use the eigenvalue/eigenvector method to solve this initial value problem:

$$\mathbf{x}' = A\mathbf{x}, \quad \text{where} \quad A = \begin{pmatrix} -\frac{2}{5} & 0 \\ \frac{2}{5} & -\frac{1}{4} \end{pmatrix}, \qquad \mathbf{x}(\mathbf{0}) \equiv \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 15 \\ 0 \end{pmatrix}.$$

Also, find the maximum value of $x_2(t)$ for $t \geq 0$.

D. **Solution:** The coefficient matrix $A = \begin{pmatrix} -\frac{2}{5} & 0 \\ \frac{2}{5} & -\frac{1}{4} \end{pmatrix}$ has characteristic equation $\lambda^2 + \frac{13}{20}\lambda + \frac{1}{10} = 0$. The eigenvalues are $-\frac{2}{5}$ and $-\frac{1}{4}$, and their eigenspaces are generated respectively by $\begin{pmatrix} 3 \\ -8 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. So the general solution is

$$x_1(t) = 3c_1 e^{-\frac{2}{5}t}, \quad x_2(t) = -8c_1 e^{-\frac{2}{5}t} + c_2 e^{-\frac{1}{4}t}.$$

The initial conditions $x_1(0) = 15$ and $x_2(0) = 0$ imply that $c_1 = 5$ and $c_2 = 40$, so the solution is

$$x_1(t) = 15e^{-\frac{2}{5}t}, \quad x_2(t) = -40e^{-\frac{2}{5}t} + 40e^{-\frac{1}{4}t}.$$

To find the maximum value of x_2, we solve the equation $x_2'(t) = 0$, obtaining $t = \frac{20}{3}\ln\frac{8}{5}$, which gives the maximum value of about 6.85.

E. **(Double Credit)** This problem involves a defective matrix $A = \begin{pmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{pmatrix}$.

You are to work out the problem entirely by hand, though of course you may use a computer to check your work. Consider the differential equation

$$\mathbf{x}' = A\mathbf{x}. \tag{1}$$

a. Find the eigenvalues of the matrix A, and identify the eigenspace of each. You will find that there is only one eigenvalue (of multiplicity 3), and it is deficient, having an eigenspace of dimension 2.

b. Find the general solution of (1), that is, find a fundamental matrix of the problem. This will require you to find one “generalized eigenvector” of the eigenvalue.

c. Fine e^{At}.

d. Solve the initial value problem with $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \\ 7 \end{pmatrix}$.

E. **Solution, Method 1:** The determinant of $A - \lambda I$ is $(1-\lambda)^3$, so 1 is the only eigenvalue. Its eigenspace consists of the vectors $\mathbf{s}$ such that $(A-I)\mathbf{s} = \mathbf{0}$, which is equivalent to the one equation $4s_1 - 3s_2 - 2s_3 = 0$. One choice of two independent eigenvectors is $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$, and we can begin writing the solution:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} e^t + c_2 \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} e^t + \ldots.$$

It remains to find a "generalized eigenvector" $\mathbf{k}$ satisfying

$$(A-I)^2\mathbf{k} = \mathbf{0} \tag{2}$$

and

$$(A-I)\mathbf{k} \neq \mathbf{0}. \tag{3}$$

Since the square of $(A-I)$ is the all-zero matrix, *every* $\mathbf{k}$ satisfies (2). One choice that also satisfies (3) is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, and

$$(A-I)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ 2 \end{pmatrix}.$$

So the general solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} e^t + c_2 \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} e^t + c_3 \left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ -4 \\ 2 \end{pmatrix} t \right] e^t.$$

We can now write down a fundamental matrix:

$$\Phi(t) = \begin{pmatrix} e^t & 3e^t & -2te^t \\ 0 & 4e^t & -4te^t \\ 2e^t & 0 & e^t + 2te^t \end{pmatrix}.$$

Then

$$\Phi(0) = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 4 & 0 \\ 2 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \Phi(0)^{-1} = \frac{1}{4}\begin{pmatrix} 4 & -3 & 0 \\ 0 & 1 & 0 \\ -8 & 6 & 4 \end{pmatrix}.$$

Thus

$$e^{At} = \Phi(t).\Phi(0)^{-1} = \begin{pmatrix} e^t + 4te^t & -3te^t & -2te^t \\ 8te^t & e^t - 6te^t & -4te^t \\ -4te^t & 3te^t & e^t + 2te^t \end{pmatrix}.$$

The solution to the IVP is

$$e^{At}\begin{pmatrix} 1 \\ 1 \\ 7 \end{pmatrix} = \begin{pmatrix} e^t - 13te^t \\ e^t - 26te^t \\ 7e^t + 13te^t \end{pmatrix}.$$

E. **Solution, Method 2:** As soon as we notice that $A - I$ is nilpotent, specifically that $(A-I)^2$ is the all-zero matrix, we know we can compute e^{At} as follows:

$$\begin{aligned} e^{At} &= e^t\left(I + (A-I)t\right) = e^t\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{pmatrix} t\right) \\ &= \begin{pmatrix} e^t + 4te^t & -3te^t & -2te^t \\ 8te^t & e^t - 6te^t & -4te^t \\ -4te^t & 3te^t & e^t + te^t \end{pmatrix}, \end{aligned}$$

which is the same as what we got above, and the solution to the IVP is obtained as before.

F. **(Double Credit)** Consider the problem $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -5 & -2 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -2 & -2 \end{pmatrix}.$$

This time, there is damping on the motion of each of the masses, and the damping constants both equal 2 pounds per foot per second. There are three eigenvalues. The eigenvalue -1 is a double eigenvalue, and its eigenspace is spanned by $\begin{pmatrix} 1 \\ -1 \\ 2 \\ -2 \end{pmatrix}$. So -1 is defective. The other two eigenvalues are $-1 + i\sqrt{5}$, with eigenspace spanned by $\begin{pmatrix} 2 + i2\sqrt{5} \\ -12 \\ -1 - i\sqrt{5} \\ 6 \end{pmatrix}$; and $-1 - i\sqrt{5}$, with eigenspace spanned by $\begin{pmatrix} 2 - i2\sqrt{5} \\ -12 \\ -1 + i\sqrt{5} \\ 6 \end{pmatrix}$. Write out the general real-valued solution. (Notice that to find a second solution related to the eigenvalue -1, it is necessary to find a generalized eigenvector. The other main part of the Exercise is to find two independent real-valued solutions related to the two complex eigenvalues.)

F. **Solution:** One way to write the general real-valued solution is as follows:

$$\begin{aligned}\mathbf{x}(t) =& c_1 \begin{pmatrix} 1 \\ -1 \\ 2 \\ -2 \end{pmatrix} e^{-t} + c_2 \left[\begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 2 \\ -2 \end{pmatrix} t \right] e^{-t} \\ &+ c_3 \begin{pmatrix} 2\cos\sqrt{5}t - 2\sqrt{5}\sin\sqrt{5}t \\ -12\cos\sqrt{5}t \\ -\cos\sqrt{5}t + \sqrt{5}\sin\sqrt{5}t \\ 6\cos\sqrt{5}t \end{pmatrix} e^{-t} + c_4 \begin{pmatrix} 2\sin\sqrt{5}t + 2\sqrt{5}\cos\sqrt{5}t \\ 12\sin\sqrt{5}t \\ -\sin\sqrt{5}t - \sqrt{5}\cos\sqrt{5}t \\ -6\sin\sqrt{5}t \end{pmatrix} e^{-t}\end{aligned}.$$

G. **(Double Credit)** In the Exercise above, the vector $\mathbf{x}(t)$ that you solve for has four components. Recall that $x_1(t)$ and $x_3(t)$ give the displacements from equilibrium of the two masses, and $x_2(t)$ and $x_4(t)$ give their respective velocities. In each of the three cases identified below, find the unique solution with the given initial values. (You will find that one of the three initial-value vectors leads to motion that is entirely non-oscillatory, and each of the other two to motion in which the two masses move always in opposite directions.)

a. $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}.$

b. $\mathbf{x}(0) = \begin{pmatrix} 2 \\ -12 \\ -1 \\ 6 \end{pmatrix}.$

c. $\mathbf{x}(0) = \begin{pmatrix} 2\sqrt{5} \\ 0 \\ -\sqrt{5} \\ 0 \end{pmatrix}.$

G. **Solution:**

a. $\mathbf{x}(t) = \begin{pmatrix} e^{-t} + te^{-t} \\ -te^{-t} \\ 2e^{-t} + 2te^{-t} \\ -2te^{-t} \end{pmatrix}.$

b. $\mathbf{x}(t) = \begin{pmatrix} 2\cos\sqrt{5}t - 2\sqrt{5}\sin\sqrt{5}t \\ -12\cos\sqrt{5}t \\ -\cos\sqrt{5}t + \sqrt{5}\sin\sqrt{5}t \\ 6\cos\sqrt{5}t \end{pmatrix} e^{-t}.$

c. $\mathbf{x}(t) = \begin{pmatrix} 2\sin\sqrt{5}t + 2\sqrt{5}\cos\sqrt{5}t \\ 12\sin\sqrt{5}t \\ -\sin\sqrt{5}t - \sqrt{5}\cos\sqrt{5}t \\ -6\sin\sqrt{5}t \end{pmatrix} e^{-t}.$

Introduction to H and I. We consider again a system of two springs, with Hooke's-Law constants k_1 and k_2, and two masses m_1 and m_2. Let c_1 and c_2 be the damping constants. Let u_1 and u_2 denote the respective displacements of the two masses. From Newton's Law and Hooke's Law we have derived the following system of two second-order equations:

$$\begin{aligned} u_1'' + \frac{c_1}{m_1}u_1' + \frac{k_1+k_2}{m_1}u_1 - \frac{k_2}{m_1}u_2 &= 0, \\ u_2'' + \frac{c_2}{m_2}u_2' - \frac{k_2}{m_2}u_1 + \frac{k_2}{m_2}u_2 &= 0. \end{aligned}$$

Letting $x_1 = u_1, x_2 = u_1', x_3 = u_2$, and $x_4 = u_2'$, we obtain an equivalent system of four first-order equations:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -\frac{k_1+k_2}{m_1}x_1 - \frac{c_1}{m_1}x_2 + \frac{k_2}{m_1}x_3 \\ x_3' &= x_4 \\ x_4' &= \frac{k_2}{m_2}x_1 - \frac{k_2}{m_2}x_3 - \frac{c_2}{m_2}x_4 \end{aligned}$$

This is $X' = AX$, where

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & -\frac{c_1}{m_1} & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & -\frac{k_2}{m_2} & -\frac{c_2}{m_2} \end{pmatrix}.$$

In the following Exercises H and I, the values $m_1 = 1$ and $m_2 = 1$ are fixed. You are welcome to do these Exercises just by experimenting, making use of the Mathematica command *Eigenvalues*[A], but you may find it helpful to do a bit of theorizing, or a bit of educated guessing.

H. **(Triple Credit)** Fix the spring constants $k_1 = 3$ and $k_2 = 2$, so that

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -5 & -c_1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -2 & -c_2 \end{pmatrix}.$$

Now: You are allowed to vary each of the damping constants c_1 and c_2 from 0.0 to 100.0 at increments of 0.1. The cost of each increment is \$100. For example, if your choice is $c_1 = 2$ and $c_2 = 3$, then the cost is \$5000. Your objective is to select the damping constants so that the cost is as little as possible, and so that no oscillatory motion occurs. In terms of the mathematical model, that means the eigenvalues of the matrix A must all be real. Report your selection and justify it by showing pertinent calculations and presenting whatever discussion is appropriate.

H. **Solution:** With $c_1 = 5.4$ and $c_2 = 4$, and also with $c_1 = 5.3$ and $c_2 = 4.1$, the eigenvalues are real and the cost is \$9400. I believe those are the only answers. Finding either of those pairs was sufficient. If your answer was more expensive, one point was subtracted for each \$100 overrun. You got no credit for the assignment if you worked with an incorrect matrix; the giveaway in such cases is a positive eigenvalue, which won't occur in a well-damped mechanical system.

I. **(Triple Credit)** This time, you are allowed to vary each of the damping constants, just as in Exercise A, and you are allowed also to vary each of the Hooke's Law constants from 1.0 to 10.0 in increments of 0.1. Whatever choice of the four constants you make, you must pay \$1000 times $k_1 + k_2 + c_1 + c_2$. For example, if $k_1 = 3$, $k_2 = 2$, and $c_1 = c_2 = 5$, the cost is $\$15,000$. Your objective is to select the four constants so that the cost is as little as possible, and so that no oscillatory motion occurs. In terms of the mathematical model, that means the eigenvalues of the matrix A must all be real. Report your selection and justify it by showing pertinent calculations and presenting whatever discussion is appropriate.

I. **Solution:** The spring constants should be taken both equal to 1 for the best result. Then the c_1, c_2 pairs that give the minimum cost of \$8400 are 3.3, 3.1; 3.4, 3.0; 3.5, 2.9; 3.6, 2.8; and 3.7, 2.7. For full credit, it sufficed to find one of them. Cost overruns were penalized by one point per \$100. No credit if you worked with an incorrect matrix.