Math 2090 (McGehee) Problem Set 6, Due Wednesday, November 10, 2004
Please fold papers the long way, put name on outside.

A. Consider the initial value problem

$$y'' + 5y' + 6y = \cos 2t, \quad y(0) = 2, \quad y'(0) = -5. \tag{1}$$

Show in detail how to find the solution by the method of Laplace Transforms. You should make your presentation neat, complete, and clear. You need not do the partial-fractions decomposition by hand if you can get a computer to do it for you.

A. **Solution:** Here's the "Laplace Transform of the problem (1):"

$$s^2Y(s) - 2s + 5 + 5(sY(s) - 2) + 6Y(s) = \frac{s}{s^2+4}, \quad \text{or}$$

$$Y(s)(s^2 + 5s + 6) = 2s + 5 + \frac{s}{s^2+4}.$$

Solving for $Y(s)$, we obtain

$$Y(s) = \frac{2s+5}{s^2+5s+6} + \frac{s}{(s^2+4)(s^2+5s+6)}.$$

The partial-fractions decompositions required are as follows:

$$\frac{2s+5}{s^2+5s+6} = \frac{1}{s+2} + \frac{1}{s+3} \quad \text{and}$$
$$\frac{s}{(s^2+4)(s^2+5s+6)} = \frac{s+10}{52(s^2+4)} - \frac{1}{4(s+2)} + \frac{3}{13(s+3)}.$$

Now apply the inverse Laplace transform:

$$\begin{aligned} y(t) &= e^{-2t} + e^{-3t} \\ &\quad + \frac{1}{52}\cos 2t + \frac{5}{52}\sin 2t - \frac{1}{4}e^{-2t} + \frac{3}{13}e^{-3t} \\ &= \frac{1}{52}(39e^{-2t} + 64e^{-3t} + \cos 2t + 5\sin 2t). \end{aligned}$$

B. Consider the initial value problem

$$y'' + 4y = \begin{cases} 1 & \text{if } \pi \le t < 2\pi, \\ 0 & \text{if } 0 \le t < \pi \text{ or } t \ge 2\pi, \end{cases}$$

where $y(0) = 1$ and $y'(0) = 0$.

a. Show in detail how to find the solution by the method of Laplace Transforms. You should make your presentation neat, complete, and clear. But you need not do the partial-fractions decomposition by hand.

b. Provide a reasonably accurate graph of the solution.

B. **Solution:** The right-hand side of the DE can be written in terms of the unit-step function. Thus the DE becomes

$$y''(t) + 4y(t) = u(t-\pi) - u(t-2\pi).$$

Take the Laplace transform:

$$s^2Y(s) - s + 4Y(s) = \frac{e^{-\pi s}}{s} - \frac{e^{-2\pi s}}{s}.$$

Solve for $Y(s)$:

$$Y(s) = \frac{s}{s^2+4} + \frac{e^{-\pi s}}{s(s^2+4)} - \frac{e^{-2\pi s}}{s(s^2+4)}.$$

The partial-fractions decomposition that we need is

$$\frac{1}{s(s^2+4)} = \frac{1}{4}\left(\frac{1}{s} - \frac{s}{s^2+4}\right),$$

which has inverse Laplace transform

$$\frac{1}{4}(1-\cos 2t).$$

Finally, then, the solution is obtained by taking the inverse Laplace transform of $Y(s)$:

$$\begin{aligned} y(t) &= \cos 2t + \frac{1}{4}(1-\cos 2t)u(t-\pi) - \frac{1}{4}(1-\cos 2t)u(t-2\pi) \\ &= \begin{cases} \cos 2t & \text{for } 0 \le t \le \pi, \\ \frac{3}{4}\cos 2t + \frac{1}{4} & \text{for } \pi \le t \le 2\pi, \\ \cos 2t & \text{for } 2\pi \le t < \infty. \end{cases} \end{aligned}$$

Notice that we've used the fact that $\cos 2(t-\pi) \equiv \cos 2t$.

C. Consider the initial value problem

$$y'' + 2y' + y = \delta(t-\pi), \quad y(0) = 0, \quad y'(0) = 1.$$

a. Show in detail how to find the solution by the method of Laplace Transforms. You should make your presentation neat, complete, and clear.

b. Find the maximum value of $|y(t)|$, and the time t at which it occurs.

C. **Solution:** Take the Laplace transform on both sides, obtaining

$$s^2Y(s) - 1 + 2sY(s) + Y(s) = e^{-\pi s}, \quad \text{or}$$

$$Y(s) = (1+e^{-\pi s})\frac{1}{(s+1)^2}.$$

Now take inverse Laplace transforms:

$$y(t) = te^{-t} + u(t-\pi)(t-\pi)e^{-(t-\pi)}.$$

The maximum on the interval $[0,\pi]$ is $\frac{1}{e} \approx .37$ and occurs when $t=1$. The maximum after $t=\pi$ occurs at about $t=4.01$ and equals about .43.

D. Consider the initial value problem

$$y'' + 2y' + y = \delta(t-\pi) + u(t-2\pi)\sin t, \quad y(0)=0, \quad y'(0)=1.$$

Show in detail how to find the solution by the method of Laplace Transforms. You should make your presentation neat, complete, and clear. But you need not do the partial-fractions decomposition by hand.

D. **Solution:** When we take the Laplace transform and then solve for it, we obtain

$$Y(s) = (1+e^{-\pi s})\frac{1}{(s+1)^2} + \frac{e^{-2\pi s}}{(s^2+1)(s+1)^2}.$$

The needed partial-fractions decomposition is

$$\frac{1}{(s^2+1)(s+1)^2} = -\frac{1}{2}\frac{s}{s^2+1} + \frac{1}{2}\frac{1}{s+1} + \frac{1}{2}\frac{1}{(s+1)^2}.$$

So

$$\begin{aligned} y(t) = te^{-t} &+ u(t-\pi)(t-\pi)e^{-(t-\pi)} \\ &+ u(t-2\pi)\left(-\frac{1}{2}\cos t + \frac{1}{2}e^{-(t-2\pi)} + \frac{1}{2}(t-2\pi)e^{-(t-2\pi)}\right). \end{aligned}$$

E. Show in detail how to compute the Laplace transform of the sawtooth function given by

$$f(t) = t \text{ for } 0 \le t < 1; \quad f(t+1) = f(t) \text{ for all } t.$$

E. **Solution:** We know that

$$F(s) = \frac{1}{1-e^{-s}}\int_0^1 e^{-st}t\,dt.$$

To evaluate that integral, we need to do integration by parts, with $u=t$ and $dv = e^{-st}\,dt$, so that $v = -\dfrac{e^{-st}}{s}$:

$$\begin{aligned} \int_0^1 e^{-st}t\,dt &= -\left.\frac{te^{-st}}{s}\right|_0^1 + \frac{1}{s}\int_0^1 e^{-st}\,dt \\ &= -\frac{e^{-s}}{s} - \left(\left.\frac{e^{-st}}{s^2}\right|_0^1\right. \\ &= -\frac{e^{-s}}{s} - \frac{e^{-s}-1}{s^2} \\ &= \frac{1-e^{-s}-se^{-s}}{s^2}. \end{aligned}$$

So

$$F(s) = \frac{1}{s^2} - \frac{e^{-s}}{s(1-e^{-s})}.$$

F. A Volterra Equation is one of the form

$$x(t) = f(t) + \int_0^t k(t-v)x(v)\,dv,$$

where f and k are given and x is the function to be solved for. See p. 575 in Section 9.9 of Goode's book. Such a problem is easily solved by taking the Laplace transform, since the integral is a convolution. Get $X(s) = F(s) + K(s)X(s)$, solve for $X(s)$, and then take the inverse Laplace transform. Solve this Volterra equation:

$$x(t) = 2e^{3t} - \int_0^t e^{2(t-v)}x(v)\,dv.$$

F. **Solution:** Taking the Laplace transform on both sides, we obtain

$$X(s) = \frac{2}{s-3} - \frac{1}{s-2}X(s),$$

from which we find that

$$X(s) = \frac{2s-4}{(s-1)(s-3)} = \frac{1}{s-1} + \frac{1}{s-3},$$

so

$$x(t) = e^t + e^{3t}.$$

G. If you are starting now to study for the final exam, then it would be good to review the material on first-order equations (Chapter 1). The following Exercise provides an occasion to review the standard method for linear first-order equations and to learn about mixing problems, which are discussed in Section 1.7 (p. 52). A tank contains (at time $t = 0$) 1000 liters of pure water. Brine containing .01 kg. of salt per liter then begins to flow into the tank at the rate of 100 liters per minute. The mix in the tank is kept uniform by stirring. Liquid flows out of the tank at the rate of 100 liters per minute. When will the concentration in the tank reach .009 kg. per liter? Show your procedure.

G. **Solution:** Let $x(t)$ be the number of kg. of salt in the tank at time t minutes. Then $x(0) = 0$. We are being asked to find the value of t such that $x(t) = .009 \times 1000 = 9$

kg. Salt enters the tank at the rate of one kg. per minute and flows out at the rate of 100 times $x(t)/1000$ kg. per minute. Thus

$$x'(t) = 1 - \frac{1}{10}x(t), \quad x(0) = 0.$$

Therefore (by the method of Section 1.5)

$$x(t) = e^{-t/10}\int_0^t e^{v/10}\,dv = e^{-t/10}10(e^{t/10} - 1) = 10(1 - e^{-t/10}),$$

which equals 9 when $e^{-t/10} = .1$ or $t = 10\ln 10 \approx 23$ minutes.

H. (Triple Credit) A large tank is partially filled with 100 gallons of a solution in which 10 pounds of salt are dissolved. Brine containing 2 pounds of salt per gallon is pumped in at 3 gallons per minute. The mix in the tank is kept uniform by stirring. The solution is pumped out at G gallons per minute. Let $u(t)$ be the number of pounds of salt in the tank at t minutes. In each of these cases, solve for the function $u(t)$, and for $0 \le t \le 100$ provide an accurate picture of its graph:

a. $G = 3$.

b. $G = 5$.

c. $G = 1$.

I. This problem provides an occasion for you to learn about Cauchy-Euler Equations, which are discussed in Section 2.9 of Goode's book. Find the general solution of this equation:

$$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = 0 \qquad (x > 0).$$

H Since we are told that the tank is "large," let's assume that the tank has infinite capacity. Let $u(t)$ be the number of pounds of salt in the tank at time t minutes. We are given that $u(0) = 10$ pounds of salt, and that there are 100 gallons of solution in the tank when $t = 0$. There is an intake and an outgo of solution. Due to the intake, 3 gallons of solution and 6 pounds of salt are entering the tank per minute. Due to the outgo, G gallons of solution and

$$\frac{Gu(t)}{100 + (3 - G)t}$$

pounds of salt are exiting the tank per minute. The differential equation is

$$u'(t) = 6 - G\frac{u(t)}{100 + (3 - G)t}, \qquad u(0) = 10,$$

which is easily put into the form of a first order linear d.e.:

$$u' + \frac{G}{100 + (3 - G)t}u = 6, \qquad u(0) = 10.$$

Before starting calculations we can make these observations: If $G = 5$, the tank is empty at 50 minutes, so that our function is of interest only on the interval $0 \le t \le 50$. In the other cases, we can expect that the limit as $t \to \infty$ of the concentraion in the tank is 2 pounds per gallon.

a. If $G = 3$, the equation becomes

$$u' + \frac{3}{100}u = 6, \qquad u(0) = 10,$$

and (omitting the details of the procedure) the solution is

$$u(t) = 200 - 190e^{-.03t}.$$

b. Before specializing to the value $G = 5$, let's find the solution for the general case in which G is positive but not equal to 3. The integrating factor is

$$\mu(t) = e^{\int \frac{G}{100+(3-G)t}dt} = e^{\frac{G}{3-G}\ln(100+(3-G)t)} = (100 + (3 - G)t)^{\frac{G}{3-G}}.$$

Thus

$$u(t) = \frac{\int 6(100 + (3 - G)t)^{\frac{G}{3-G}}dt + c}{(100 + (3 - G)t)^{\frac{G}{3-G}}} = \frac{2(100 + (3 - G)t)^{\left(\frac{G}{3-G}+1\right)} + c}{(100 + (3 - G)t)^{\frac{G}{3-G}}},$$

and finally

$$u(t) = 2(100 + (3 - G)t) + c(100 + (3 - G)t)^{\frac{G}{G-3}}.$$

It remains to plug in the value of G and to determine c by using the fact that $u(0) = 10$. Doing so, with $G = 5$, we obtain

$$u(t) = 200 - 4t - .0019(100 - 2t)^{5/2} \qquad (0 \le t \le 50).$$

c. With $G = 1$, we get

$$u(t) = 200 + 4t - 1900(100 + 2t)^{-1/2} \qquad (0 \le t < \infty).$$

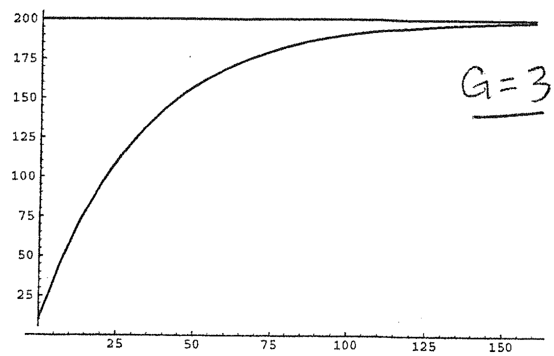
It is easy to confirm that

$$\lim_{t\to\infty} \frac{u(t)}{100 + 2t} = 2.$$

Graphs for this problem appear in the addendum.

□ Problem 4. Here are plots of the 3 solutions. The horizontal asymptote is drawn in the first one.

```
Plot[{200 - 190 E^(-.03 t),200},{t,0,160},
AxesOrigin->{0,0},PlotRange->All]
```



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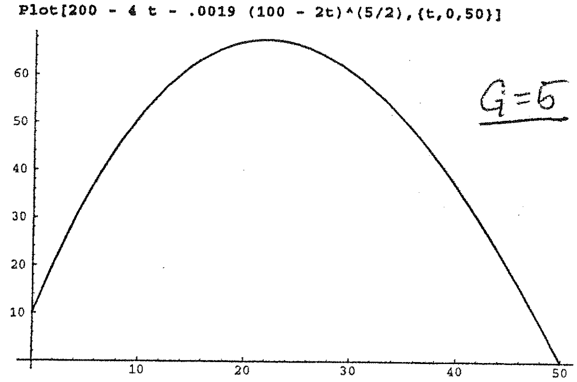
```
Plot[200 - 4 t - .0019 (100 - 2t)^(5/2),{t,0,50}]
```



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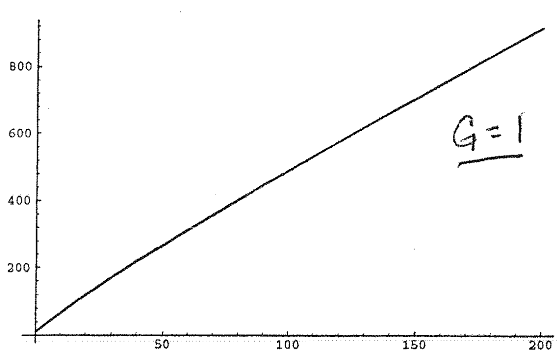
```
Plot[200 + 4 t - 1900(100 + 2t)^(-1/2),{t,0,200}]
```



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The last graph asymptotically approaches a straight line with slope 4. Of course, it is unrealistic, since the tank can't really be infinite.

```
.0019*N[2^(5/2)]
{0.010748}
```