Math 2090 (McGehee) Notes on PS7 11/26/04

- The problem is x' = ps.x , where ps = $\begin{pmatrix} 0 & 1 & 0 & 0 \\ -5 & -2 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -2 & -2 \end{pmatrix}$.

The eigenvalues of ps are -1 , -1 , $-1 + \iota\sqrt{5}$, and $-1 - \iota\sqrt{5}$. The eigenspace belonging to -1 is one-dimensional and is spanned by $\begin{pmatrix} -1 \\ 1 \\ -2 \\ 2 \end{pmatrix}$. Thus one solution of the DE will be $\begin{pmatrix} -1 \\ 1 \\ -2 \\ 2 \end{pmatrix} e^{-t}$.
Since -1 is a double root of the characteristic polynomial of ps , we must look for a second independent solution of the form $(k + (ps + I)\, kt)\, e^{-t}$, where k is a vector satisfying $(ps + I)^2\, k = 0$ and (ps+I)k ≠ 0. (For other examples of this procedure, see the lecture of October ~~15~~ 17 and Goode, ~~p. 434~~ §8.6.) Let's compute the square of ps+I and identify its kernel:

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pspi = (ps + IdentityMatrix[4]) . (ps + IdentityMatrix[4])
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$$\begin{pmatrix} -4 & 0 & 2 & 0 \\ 0 & -4 & 0 & 2 \\ 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \end{pmatrix}$$

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k = {a, b, c, d}; Solve[pspi . k == 0, {a, b, c, d}]
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$$\left\{\left\{a \to \frac{c}{2},\ b \to \frac{d}{2}\right\}\right\}$$

- Thus the kernel can be represented as $C_1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}$. Any choice of k from this kernel will give us an equivalent result. Let's take k = $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}$. Then

$$\text{(ps + IdentityMatrix[4])} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \\ -2 \end{pmatrix}$$

- So $\left(\begin{pmatrix}0\\1\\0\\2\end{pmatrix}+\begin{pmatrix}1\\-1\\2\\-2\end{pmatrix}t\right)e^{-t}$ is a second independent solution. Now let's look at the third eigenvalue and its eigenvector, writing every complex number in the form $x + \iota y$, and then writing the complex − valued vector solution in the same way:

$$\begin{pmatrix}2+i\,2\sqrt{5}\\-12\\-1-i\sqrt{5}\\6\end{pmatrix}e^{-t}\left(\cos\sqrt{5}\,t+\iota\sin\sqrt{5}\,t\right) =$$

$$e^{-t}\begin{pmatrix}2\cos\sqrt{5}\,t-2\sqrt{5}\sin\sqrt{5}\,t\\-12\cos\sqrt{5}\,t\\-\cos\sqrt{5}\,t+\sqrt{5}\sin\sqrt{5}\,t\\6\cos\sqrt{5}\,t\end{pmatrix}+\iota e^{-t}\begin{pmatrix}2\sin\sqrt{5}\,t+2\sqrt{5}\cos\sqrt{5}\,t\\-12\sin\sqrt{5}\,t\\-\sqrt{5}\cos\sqrt{5}\,t-\sin\sqrt{5}\,t\\6\sin\sqrt{5}\,t\end{pmatrix}.$$

If we worked with the third eigenvalue / eigenvector pair, we would get the same two vector − valued functions with a minus sign between them. Thus we've found two independent real − valued functions, and finally we're ready to write down the general real − valued solution:

$$x(t) = c_1\begin{pmatrix}-1\\1\\-2\\2\end{pmatrix}e^{-t} + c_2\left(\begin{pmatrix}0\\1\\0\\2\end{pmatrix}+\begin{pmatrix}1\\-1\\2\\-2\end{pmatrix}t\right)e^{-t} + c_3 e^{-t}\begin{pmatrix}2\cos\sqrt{5}\,t-2\sqrt{5}\sin\sqrt{5}\,t\\-12\cos\sqrt{5}\,t\\-\cos\sqrt{5}\,t+\sqrt{5}\sin\sqrt{5}\,t\\6\cos\sqrt{5}\,t\end{pmatrix}+$$

$$c_4 e^{-t}\begin{pmatrix}2\sin\sqrt{5}\,t+2\sqrt{5}\cos\sqrt{5}\,t\\-12\sin\sqrt{5}\,t\\-\sqrt{5}\cos\sqrt{5}\,t-\sin\sqrt{5}\,t\\6\sin\sqrt{5}\,t\end{pmatrix}.$$

A close inspection reveals that thesolutions of the initial value problems, parts 2 and 3 of the assignment, are precisely the second and third solution vectors in this sum of four. But those answers are also obtainable, for example, by the method of Laplace transforms; Bob Nystrom's write − up appears below. I chose the two initial − value vectors so that they would produce precisely those motions of the system in which (1) the two masses are always moving in the same direction and (2) they are always moving in opposite directions.

BOB NYSTROM

HOMEWORK #4

② FIND $x(t)$, GIVEN $x' = Ax$, $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -5 & -2 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -2 & -2 \end{pmatrix}$, $x(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}$

$$x' = Ax$$
$$L[x'] = L[Ax]$$
$$-x(0) + sX(s) = AX(s)$$
$$X(s)(sI - A) = x(0)$$
$$X(s) = (sI-A)^{-1} x(0)$$
$$L^{-1}[X(s)] = L^{-1}[(sI-A)^{-1} x(0)]$$
$$x(t) = L^{-1}[(sI-A)^{-1} x(0)]$$

$$L^{-1}\left[\frac{1}{(s+1)^2}\right] = L^{-1}\left[\frac{1!}{(s-(-1))^{1+1}}\right] = e^{-t}t$$

$$L^{-1}\left[\frac{1}{s+1}\right] = L^{-1}\left[\frac{1}{s-(-1)}\right] = e^{-t}$$

FROM MATHEMATICA →

$$= L^{-1}\left[\begin{pmatrix} \frac{1}{(s+1)^2} \\ \frac{1}{s+1} - \frac{1}{(s+1)^2} \\ \frac{2}{(s+1)^2} \\ \frac{2}{s+1} - \frac{2}{(s+1)^2} \end{pmatrix}\right] = \boxed{\begin{bmatrix} e^{-t}t \\ e^{-t} - e^{-t}t \\ 2e^{-t}t \\ 2e^{-t} - 2e^{-t}t \end{bmatrix} = x(t)}$$

BOB NYSTROM

HOMEWORK #4

③ FIND $x(t)$, GIVEN $x' = Ax$, $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -5 & -2 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -2 & 2 \end{pmatrix}$, $x(0) = \begin{pmatrix} 2 \\ -12 \\ -1 \\ 6 \end{pmatrix}$

$$x' = Ax$$
$$L[x'] = L[Ax]$$
$$-x(0) + sX(s) = AX(s)$$
$$X(s) = (sI - A)^{-1} x(0)$$
$$x(t) = L^{-1}\left[(sI-A)^{-1} x(0)\right]$$

FROM MATHEMATICA →

$$= L^{-1}\left[\begin{pmatrix} \frac{2(s-4)}{s^2+2s+6} \\ -\frac{12(s+1)}{s^2+2s+6} \\ \frac{4-s}{s^2+2s+6} \\ \frac{6(s+1)}{s^2+2s+6} \end{pmatrix}\right]$$

$$2L^{-1}\left[\frac{s-4}{s^2+2s+6}\right] = 2L^{-1}\left[\frac{s+1}{(s+1)^2+(\sqrt{5})^2} + (-\sqrt{5})\frac{\sqrt{5}}{(s+1)^2+(\sqrt{5})^2}\right]$$
$$= 2e^{-t}\cos\sqrt{5}t - 2\sqrt{5}e^{-t}\sin\sqrt{5}t$$

$$-12L^{-1}\left[\frac{s+1}{s^2+2s+6}\right] = -12L^{-1}\left[\frac{s-(-1)}{(s-(-1))^2+(\sqrt{5})^2}\right]$$
$$= -12e^{-t}\cos\sqrt{5}t$$

$$L^{-1}\left[\frac{4-s}{s^2+2s+6}\right] = L^{-1}\left[(\sqrt{5})\frac{\sqrt{5}}{(s+1)^2+(\sqrt{5})^2} - \frac{s+1}{(s+1)^2+(\sqrt{5})^2}\right]$$
$$= \sqrt{5}e^{-t}\sin\sqrt{5}t - e^{-t}\cos\sqrt{5}t$$

$$6L^{-1}\left[\frac{s+1}{s^2+2s+6}\right] = 6L^{-1}\left[\frac{s-(-1)}{(s-(-1))^2+(\sqrt{5})^2}\right]$$
$$= 6e^{-t}\cos\sqrt{5}t$$

$$x(t) = \begin{pmatrix} 2e^{-t}\cos\sqrt{5}t - 2\sqrt{5}e^{-t}\sin\sqrt{5}t \\ -12e^{-t}\cos\sqrt{5}t \\ \sqrt{5}e^{-t}\sin\sqrt{5}t - e^{-t}\cos\sqrt{5}t \\ 6e^{-t}\cos\sqrt{5}t \end{pmatrix}$$

Math 2090 (McGehee) Problem Set 7 Due 11/24/04 (Two Masses, Two Springs)

- **Consider the problem x' = ps.x , where ps is the 4 by 4 matrix given below. You may use the eigenvector-eigenvalue methods of Chapter 8 [redacted] or the Laplace transform methods presented in class, or both. Write up your work with clear and thorough explanations.**

TRIPLE CREDIT **(1) Find the general real-valued solution.**

'' **(2) Find the solution of the initial value problem x' = ps.x, x(0) =** $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}$.

'' **(3) Find the solution of the initial value problem x' = ps.x, x(0) =** $\begin{pmatrix} 2 \\ -12 \\ -1 \\ 6 \end{pmatrix}$.

In[2]:= $\texttt{ps} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -5 & -2 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -2 & -2 \end{pmatrix};$

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Eigensystem[ps]
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Out[3]= $\begin{pmatrix} -1 & -1 & -1 - i\sqrt{5} & -1 + i\sqrt{5} \\ \{-1, 1, -2, 2\} & \{0, 0, 0, 0\} & \left\{-\frac{2i}{-i+\sqrt{5}}, -2, \frac{i}{-i+\sqrt{5}}, 1\right\} & \left\{\frac{2i}{i+\sqrt{5}}, -2, -\frac{i}{i+\sqrt{5}}, 1\right\} \end{pmatrix}$

- ***Mathematica* has chosen to write some of the eigenvectors in a form you may dislike. Another way to give the same information: The eigenspace belonging to the eigenvalue $-1 \pm i\sqrt{5}$ is spanned by the vector** $\begin{pmatrix} 2 \pm i\,2\sqrt{5} \\ -12 \\ -1 \mp i\sqrt{5} \\ 6 \end{pmatrix}$.

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In[7]:= psa = s IdentityMatrix[4] - ps
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Out[7]= $\begin{pmatrix} s & -1 & 0 & 0 \\ 5 & s+2 & -2 & 0 \\ 0 & 0 & s & -1 \\ -2 & 0 & 2 & s+2 \end{pmatrix}$

In[8]:= **psainv = Inverse[psa]**

Out[8]=
$$\begin{pmatrix} \frac{s^3+4s^2+6s+4}{s^4+4s^3+11s^2+14s+6} & \frac{s^2+2s+2}{s^4+4s^3+11s^2+14s+6} & \frac{2s+4}{s^4+4s^3+11s^2+14s+6} & \frac{2}{s^4+4s^3+11s^2+14s+6} \\ \frac{-5s^2-10s-6}{s^4+4s^3+11s^2+14s+6} & \frac{s^3+2s^2+2s}{s^4+4s^3+11s^2+14s+6} & \frac{2s^2+4s}{s^4+4s^3+11s^2+14s+6} & \frac{2s}{s^4+4s^3+11s^2+14s+6} \\ \frac{2s+4}{s^4+4s^3+11s^2+14s+6} & \frac{2}{s^4+4s^3+11s^2+14s+6} & \frac{s^3+4s^2+9s+10}{s^4+4s^3+11s^2+14s+6} & \frac{s^2+2s+5}{s^4+4s^3+11s^2+14s+6} \\ \frac{2s^2+4s}{s^4+4s^3+11s^2+14s+6} & \frac{2s}{s^4+4s^3+11s^2+14s+6} & \frac{-2s^2-4s-6}{s^4+4s^3+11s^2+14s+6} & \frac{s^3+2s^2+5s}{s^4+4s^3+11s^2+14s+6} \end{pmatrix}$$

Apart[psainv . $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$]

$$\begin{pmatrix} \frac{4(s+2)}{5(s^2+2s+6)}+\frac{1}{5(s+1)}+\frac{1}{5(s+1)^2} \\ -\frac{1}{5(s+1)^2}-\frac{24}{5(s^2+2s+6)} \\ -\frac{2(s+2)}{5(s^2+2s+6)}+\frac{2}{5(s+1)}+\frac{2}{5(s+1)^2} \\ \frac{12}{5(s^2+2s+6)}-\frac{2}{5(s+1)^2} \end{pmatrix}$$

Apart[psainv . $\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$]

$$\begin{pmatrix} \frac{1}{5(s+1)^2}+\frac{4}{5(s^2+2s+6)} \\ \frac{4s}{5(s^2+2s+6)}+\frac{1}{5(s+1)}-\frac{1}{5(s+1)^2} \\ \frac{2}{5(s+1)^2}-\frac{2}{5(s^2+2s+6)} \\ -\frac{2s}{5(s^2+2s+6)}+\frac{2}{5(s+1)}-\frac{2}{5(s+1)^2} \end{pmatrix}$$

Apart[psainv . $\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$]

$$\begin{pmatrix} -\frac{2(s+2)}{5(s^2+2s+6)}+\frac{2}{5(s+1)}+\frac{2}{5(s+1)^2} \\ \frac{12}{5(s^2+2s+6)}-\frac{2}{5(s+1)^2} \\ \frac{s+2}{5(s^2+2s+6)}+\frac{4}{5(s+1)}+\frac{4}{5(s+1)^2} \\ -\frac{4}{5(s+1)^2}-\frac{6}{5(s^2+2s+6)} \end{pmatrix}$$

Apart[psainv . $\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$]

$$\begin{pmatrix} \frac{2}{5(s+1)^2}-\frac{2}{5(s^2+2s+6)} \\ -\frac{2s}{5(s^2+2s+6)}+\frac{2}{5(s+1)}-\frac{2}{5(s+1)^2} \\ \frac{4}{5(s+1)^2}+\frac{1}{5(s^2+2s+6)} \\ \frac{s}{5(s^2+2s+6)}+\frac{4}{5(s+1)}-\frac{4}{5(s+1)^2} \end{pmatrix}$$

In[9]:= **Apart[psainv . $\begin{pmatrix}0\\1\\0\\2\end{pmatrix}$]**

Out[9]=
$$\begin{pmatrix} \frac{1}{(s+1)^2} \\ \frac{1}{s+1}-\frac{1}{(s+1)^2} \\ \frac{2}{(s+1)^2} \\ \frac{2}{s+1}-\frac{2}{(s+1)^2} \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ -12 \\ -1 \\ 6 \end{pmatrix}$$

In[10]:= **Apart[psainv . $\begin{pmatrix}2\\-12\\-1\\6\end{pmatrix}$]**

Out[10]=
$$\begin{pmatrix} \frac{2(s-4)}{s^2+2s+6} \\ -\frac{12(s+1)}{s^2+2s+6} \\ \frac{4-s}{s^2+2s+6} \\ \frac{6(s+1)}{s^2+2s+6} \end{pmatrix}$$