Math 2090 (McGehee) Test 2 October 27, 2004

This is a closed-book, closed-notes test. No computers, calculators, or cell phones. This is a 90-minute test. Show your procedures clearly and completely. You may make use of the computations at the end of the test.

1. Consider an arbitrary 3×3 matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.

   a. Find a matrix B such that $BA = \begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.

   b. Find a matrix C such that $CA = \begin{pmatrix} a_{11} + 2a_{31} & a_{12} + 2a_{32} & a_{13} + 2a_{33} \\ a_{21} & a_{22} & a_{23} \\ 3a_{31} & 3a_{32} & 3a_{33} \end{pmatrix}$.

   c. If the determinant of A is Det $(A) = 5$, then Det $(BA) =$? and Det $CA =$?.

2. The eigenvalues of the matrix $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -2 & 0 \\ 0 & -5 & 2 \end{pmatrix}$ are 1, 2, and -2.

   a. Show how to find the eigenspace of each eigenvalue. In other words, find the set of eigenvectors belong to each of the three eigenvalues.

   b. Find the general solution of the problem $\mathbf{x}' = A\mathbf{x}$.

   c. Find a fundamental matrix for the problem.

   d. Find the transition matrix e^{At}.

   e. Find a matrix S such that $S^{-1}AS$ is a diagonal matrix.

3. Find an example of one 4×4 matrix A, which of course defines a linear map from $\mathbf{R}^4$ to $\mathbf{R}^4$, such that these two conditions are both satisfied:

   (a) The vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ are in the range of the map; and

   (b) The vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ are in the nullspace (kernel) of the map.

4. Let $A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}$.

   a. Find the eigenvalues of A. Show your procedure.

   b. Find the general real-valued solution of the problem $\mathbf{x}' = A\mathbf{x}$.

   c. Find the transition matrix e^{At}.

5. Let $A = \begin{pmatrix} 6 & -8 \\ 2 & -2 \end{pmatrix}$.

   a. Find the eigenvalues and eigenvectors of the matrix A.

   b. Find the general solution of the problem $\mathbf{x}' = A\mathbf{x}$.

   c. Solve the initial value problem $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Some Computations

(a) $\begin{pmatrix} 1 & 2 & -1 \\ 0 & -2 & 0 \\ 0 & -5 & 2 \end{pmatrix} . \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 4 \\ 0 & -1 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 0 & -8 \\ 0 & 2 & -10 \end{pmatrix}.$

(b) $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix} . \begin{pmatrix} -i & +i & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ +i & -i & 0 \\ 0 & 0 & 5 \end{pmatrix}.$

(c) **Inverse** $\left[\begin{pmatrix} 1 & 3 & 1 \\ 3 & -1 & 1 \\ 0 & 0 & 3 \end{pmatrix} \right] = \begin{pmatrix} \frac{1}{10} & \frac{3}{10} & -\frac{2}{15} \\ \frac{3}{10} & -\frac{1}{10} & -\frac{1}{15} \\ 0 & 0 & \frac{1}{3} \end{pmatrix}.$

(d) $\begin{pmatrix} 1 & 3 & -1 \\ 3 & -1 & -1 \\ 0 & 0 & -3 \end{pmatrix} . \begin{pmatrix} \frac{1}{10} & \frac{3}{10} & -\frac{2}{15} \\ \frac{3}{10} & -\frac{1}{10} & -\frac{1}{15} \\ 0 & 0 & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\frac{2}{3} \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & -1 \end{pmatrix}.$

(e) $\begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 4 \\ 0 & -1 & 5 \end{pmatrix} . \begin{pmatrix} 1 & -1 & 1 \\ 0 & \frac{5}{4} & -1 \\ 0 & \frac{1}{4} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$

(f) $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$

(g) $\begin{pmatrix} 6 & -8 \\ 2 & -2 \end{pmatrix} . \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}.$