Math 2090 (McGehee) Test 2 Key 10/27/04

1. $B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $C = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ Det $BA = -5$. Det $CA = 15$.

2. See below. See (a), (e) on the back of the test.

3. One possibility: $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

4. $\det(A-\lambda I) = \det\begin{pmatrix} -\lambda & 1 & 0 \\ -1 & -\lambda & 0 \\ 0 & 0 & 5-\lambda \end{pmatrix} = (5-\lambda)(\lambda^2+1) = 0$ when

$\lambda = 5, i, -i$. $\underline{\lambda=5}$: $\begin{pmatrix} -5 & 1 & 0 & 0 \\ -1 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\underline{\text{Eigenspace}}$: $c_1\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\underline{\lambda=i}$: $\begin{pmatrix} -i & 1 & 0 & 0 \\ -1 & -i & 0 & 0 \\ 0 & 0 & 5-i & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ Eigenspace: $c_2\begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} e^{it} = \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}(\cos t + i\sin t) = \begin{pmatrix} \sin t \\ \cos t \\ 0 \end{pmatrix} + i\begin{pmatrix} -\cos t \\ \sin t \\ 0 \end{pmatrix}$$

General real-valued solution:

$$x = c_1\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{5t} + c_2\begin{pmatrix} \sin t \\ \cos t \\ 0 \end{pmatrix} + c_2\begin{pmatrix} -\cos t \\ \sin t \\ 0 \end{pmatrix}$$

A fundamental matrix is

$$fm(t) = \begin{pmatrix} \sin t & -\cos t & 0 \\ \cos t & \sin t & 0 \\ 0 & 0 & e^{5t} \end{pmatrix}; \quad fm(0) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$fm(0)^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad e^{At} = fm(t)\,fm(0)^{-1} = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & e^{5t} \end{pmatrix}.$$

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#2

Let $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -2 & 0 \\ 0 & -5 & 2 \end{pmatrix}$ By hand:

a. show that the eigenvalues of A are 1, 2, and -2

b. For each of the three eigenvalues, identify the eigenspace

c. Find the general solution of $x' = Ax$

$$(A - \lambda I)x = 0$$

$$\det(A - \lambda I) = 0$$

$$\det\begin{pmatrix} 1-\lambda & 2 & -1 \\ 0 & -2-\lambda & 0 \\ 0 & -5 & 2-\lambda \end{pmatrix} = (1-\lambda)(-2-\lambda)(2-\lambda) = 0$$

$$1-\lambda=0 \quad -2-\lambda=0 \quad 2-\lambda=0$$

eigenvalues $\longrightarrow$ $\lambda=1 \quad \lambda=-2 \quad \lambda=2$

The eigenspace for $\lambda = 1$

$$\underbrace{\begin{pmatrix} 0 & 2 & -1 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & -5 & 1 & 0 \end{pmatrix}}_{A-\lambda I} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$x_2 = 0$
$x_3 = 0$

The eigenspace is $k\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

for $\lambda = 2$

$$\begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & -5 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$x_1 = -x_3$
$x_2 = 0$

The eigenspace is $k\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

for $\lambda = -2$

$$\begin{pmatrix} 3 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -5 & 4 & 0 \end{pmatrix} = \begin{pmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -\frac{4}{5} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -\frac{4}{5} & 0 \end{pmatrix}$$

$x_1 = -\frac{1}{5}x_3$
$x_2 = \frac{4}{5}x_3$
$x_1 = -\frac{1}{4}x_2$

The eigenspace is $k\begin{pmatrix} -1 \\ 4 \\ 5 \end{pmatrix}$

Now to find the general solution of $x' = Ax$ in form

The eigensystem is

$$1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad 2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad -2 \begin{pmatrix} -1 \\ 4 \\ 5 \end{pmatrix}$$

with $1, 2, -2$ being eigenvectors and $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \\ 5 \end{pmatrix}$ being their respective eigenvectors

The general solution of $x' = Ax$ is

$$c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} -1 \\ 4 \\ 5 \end{pmatrix} e^{-2t}$$

The fundamental matrix $X(t)$ is

$$X(t) = \begin{pmatrix} e^{t} & e^{2t} & -e^{-2t} \\ 0 & 0 & 4e^{-2t} \\ 0 & -e^{2t} & 5e^{-2t} \end{pmatrix} \qquad X(0) = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 4 \\ 0 & -1 & 5 \end{pmatrix}$$

$$X(0)^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & \frac{5}{4} & -1 \\ 0 & \frac{1}{4} & 0 \end{pmatrix}$$

Then $e^{At} = X(t)(X(0)^{-1}) = \begin{pmatrix} e^{t} & e^{2t} & -e^{-2t} \\ 0 & 0 & 4e^{-2t} \\ 0 & -e^{2t} & 5e^{-2t} \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 0 & \frac{5}{4} & -1 \\ 0 & \frac{1}{4} & 0 \end{pmatrix}$

$$= \begin{pmatrix} e^{t} & -e^{t} + \frac{5}{4}e^{2t} - \frac{1}{4}e^{2t} & e^{t} - e^{2t} \\ 0 & e^{-2t} & 0 \\ 0 & -\frac{5}{4}e^{t} + \frac{5}{4}e^{-2t} & e^{2t} \end{pmatrix}$$

Since $x = e^{kAt}$ is a solution to $x' = Ax$

the general solution is

$$x = \vec{k} \begin{pmatrix} e^{t} & -e^{t} + \frac{5}{4}e^{2t} \frac{1}{4}e^{2t} & e^{t} - e^{2t} \\ 0 & e^{-2t} & 0 \\ 0 & -\frac{5}{4}e^{t} + \frac{5}{4}e^{-2t} & e^{2t} \end{pmatrix}$$

5. $A = \begin{pmatrix} 6 & -8 \\ 2 & -2 \end{pmatrix}$. Find eigenvalues: $\det(A - \lambda I) = (6-\lambda)(-2-\lambda) + 16$
$= (\lambda - 6)(\lambda + 2) + 16 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$. So 2 is an eigenvalue of multiplicity 2. Let's find its eigenspace: $\left(\begin{array}{cc|c} 4 & -8 & 0 \\ 2 & -4 & 0 \end{array}\right)$: $x_1 - 2x_2 = 0$.
The eigenspace is $c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$; the matrix is defective.
$(A - 2I)^2 = \begin{pmatrix} 4 & -8 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} 4 & -8 \\ 2 & -4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. We seek $\vec{k}$, not in the eigenspace of 2, such that $(A - 2I)^2 \vec{k} = \vec{0}$. One choice is $\vec{k} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$$e^{2t}\left(\vec{k} + (A - 2I)\vec{k}\,t\right) = e^{2t}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix} t\right) = \begin{pmatrix} e^{2t}(1+4t) \\ 2te^{2t} \end{pmatrix}.$$

General solution: $c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} e^{2t}(1+4t) \\ 2te^{2t} \end{pmatrix} = \vec{x}(t)$.

If $\vec{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then $\left.\begin{array}{l} 2c_1 + c_2 = 1 \\ c_1 = 0 \end{array}\right\}$, so $\vec{x}(t) = \begin{pmatrix} e^{2t}(1+4t) \\ 2te^{2t} \end{pmatrix}$.