Mathematics 2090 (McGehee) **Two Masses, Two Springs**

The system of two second-order equations:

$$\begin{aligned} u_1'' + \frac{c_1}{m_1}u_1' + \frac{k_1+k_2}{m_1}u_1 - \frac{k_2}{m_1}u_2 &= 0, \\ u_2'' + \frac{c_2}{m_2}u_2' - \frac{k_2}{m_2}u_1 + \frac{k_2}{m_2}u_2 &= 0. \end{aligned}$$

Letting $x_1 = u_1, x_2 = u_1', x_3 = u_2$, and $x_4 = u_2'$, we obtain an equivalent system of four first-order equations:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -\frac{k_1+k_2}{m_1}x_1 - \frac{c_1}{m_1}x_2 + \frac{k_2}{m_1}x_3 \\ x_3' &= x_4 \\ x_4' &= \frac{k_2}{m_2}x_1 - \frac{k_2}{m_2}x_3 - \frac{c_2}{m_2}x_4 \end{aligned}$$

This is $X' = AX$, where

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & -\frac{c_1}{m_1} & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & -\frac{k_2}{m_2} & -\frac{c_2}{m_2} \end{pmatrix}.$$

Some specific examples: First, a case in which there's no damping, with $k_1 = 3, k_2 = 2, m_1 = m_2 = 1$:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -5 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -2 & 0 \end{pmatrix}.$$

Second, another case with no damping: $k_1 = 3, k_2 = 2, m_1 = m_2 = 1/4$:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -20 & 0 & 8 & 0 \\ 0 & 0 & 0 & 1 \\ 8 & 0 & -8 & 0 \end{pmatrix}.$$

Third, like the first case but with damping constants $c_1 = c_2 = 2$:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -5 & -2 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -2 & -2 \end{pmatrix}.$$