Khovanov Homology on Quasi-Alternating Links

Moshe Cohen
math.lsu.edu/∼moshe

Department of Mathematics
Louisiana State University

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Outline

1. Quasi-Alternating Links
   - Resolutions and the Tait Checkerboard Graph
   - The Determinant and Signature
   - Recursive Definition of Quasi-Alternating
2. Theorems
   - The Matrix Tree Theorem
   - Determinant à la Quasi-trees
   - Quasi-Alternating Links are Homologically Thin

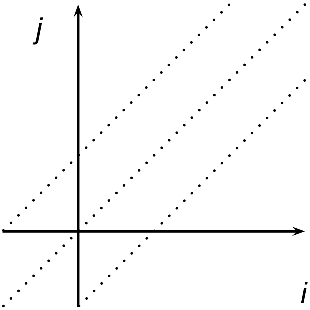
The Jones Polynomial

Reduced Khovanov Homology categorifies the Jones poly:

$$V_L(q) = \sum_{i \in \mathbb{Z}, j \in \mathbb{Z} + \frac{\ell - 1}{2}} (-1)^i q^j \operatorname{rank} \widetilde{Kh}^{i,j}(L)$$

where ℓ is the number of components of L.

The original i, j bigradings of Khovanov



An arbitrarily assigned reduced Khovanov Homology

Resolutions

As per the Kauffman Skein Relation, we'll be interested in the following two (unoriented) resolutions:

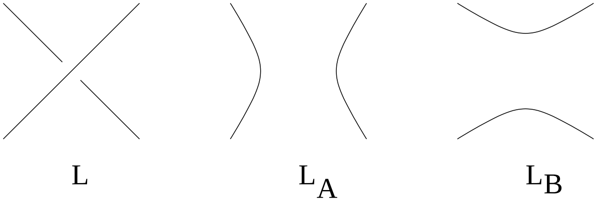


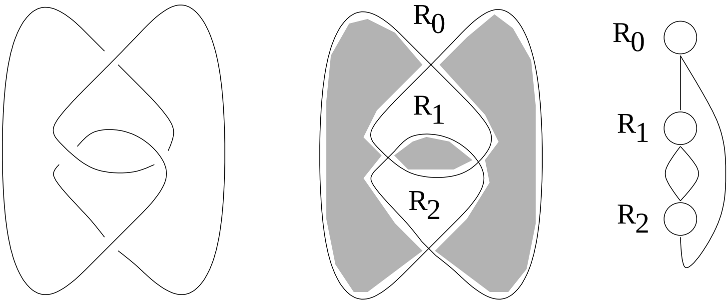
Figure: A-and B-resolutions of a crossing

More with the Tait Checkerboard Graph

Recall: We consider D as a 4-valent planar graph Γ_D and 2-color the faces black and white.

We now create the ***white* Tait checkerboard graph** Γ of D.

The white Tait checkerboard graph: an example



The 4-crossing trefoil D, the 2-colored graph Γ_D, and the desired white Tait graph Γ.

The vertices of Γ are the *white* faces of Γ_D labelled $R_0, \dots, R_n$.
The edges of Γ are the crossings c of D labelled by the incidence number $\mu(c)$ and type (I or II).

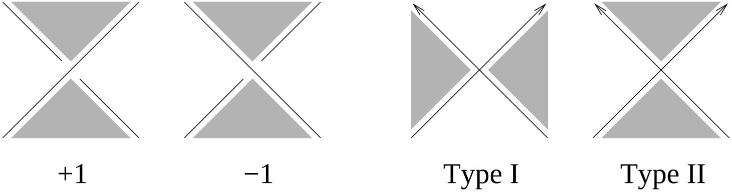
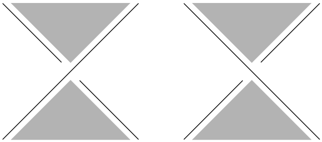


Figure: Incidence $\mu(c)$ and Type

We ensure that Γ is loopless by applying RM1 moves.



$+1$ -1

Figure: Incidence $\mu(c)$

Define the **Goeritz matrix** $G = G(D) = (g_{ij})$ for any $i \neq j \in [n]$,

$$g_{ij} = -\sum_{c=R_iR_j} \mu(c) \quad \text{and} \quad g_{ii} = -\sum_{i \neq j} g_{ij}.$$

Define the **determinant** $\det(L) = |\det(G)|$.

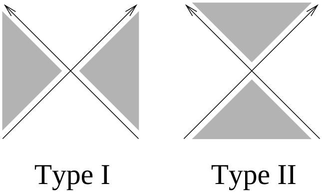


Figure: Type

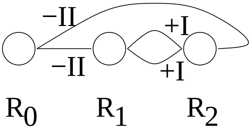
$$\text{Let} \quad \mu(D) = \sum_{c \text{ of type II}} \mu(c).$$

Gordon and Litherland showed that the **signature** $\sigma(L)$ of L is

$$\sigma(L) = \text{signature}(G) - \mu(D). \tag{1}$$

The Determinant and Signature of our example

Recall the white Tait graph obtained from the 4-crossing trefoil:



Then $G = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$ with $\det(L) = |\det(G)| = 3$;

since signature(G)=0 and $\mu(D) = -2$, we have $\sigma(L) = 2$.

Generalizing Alternating Links

Definition. Let $\mathcal{Q}$ be the set of **quasi-alternating links**, the smallest set of links satisfying:

- The unknot $\bigcirc$ is in $\mathcal{Q}$.
- If L is a link which admits a projection with a crossing such that
  1. both resolutions L_A and L_B at that crossing are in $\mathcal{Q}$,
  2. $\det(L) = \det(L_A) + \det(L_B)$,

  then L is in $\mathcal{Q}$.

Some notes on Quasi-Alternating Links

Note. 83 of the 85 prime knots with up to nine crossings are quasi-alternating. 74 are alternating.

Proposition. Non-split alternating links are quasi-alternating.

Kirchhoff's Matrix Tree Theorem (for this case). The determinant $|\det(G)|$ counts the number of spanning trees of Γ.

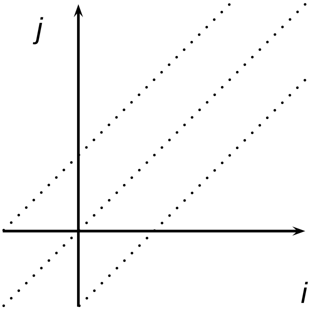
Counting Quasi-Trees with $\det(L)$

Theorem (*Dasbach, Futer, Kalfagianni, Lin, Stoltzfus*).
Consider the all-A dessin $\mathbb{D}$ associated with the (connected) diagram of L; take the genus $g(\mathbb{D})$ of the associated surface; and let $s(j, \mathbb{D})$ be the number of spanning j-quasi-trees of $\mathbb{D}$. Then

$$\det(L) = \left| \sum_{j=0}^{g(\mathbb{D})} (-1)^j s(j, \mathbb{D}) \right| .$$

Homologically Thin

Recall the arbitrary example of reduced Khovanov Homology:



Let $\delta = j - i$. A link is **homologically thin** if its homologies over R are free R-modules supported in only one δ-grading.

Main Theorem

Thm (*Lee*). Non-split alternating links are homologically thin.

Main Theorem (*Manolescu, Ozsváth*). Quasi-alternating links are Khovanov homologically thin (over $R = \mathbb{Z}$) on $\delta = \sigma/2$, where σ is the signature of the link.

Smoke and Mirrors

We have $\det(L) \geq 0$; inequality holds for quasi-alternating links.

Proof. The unknot is homologically thin on $\delta = \sigma/2$, and if both L_A and L_B are, as well, then so is L. For this we'll need the exact triangle found at the end of the talk.

□

Establishing Notation

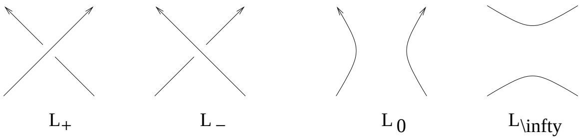


Figure: Two possible crossings and their resolutions

Fix a crossing c_0; if it is positive/negative, set $L_+/L_- = L$ and let L_-/L_+ be obtained by switching the crossing. Denote by D_* the planar diagram of L_*.

Technical Lemma

Lemma (*Murasugi; Manolescu, Ozsváth*). Suppose that $\det(L_0), \det(L_\infty) \neq 0$ and $\det(L_+) = \det(L_0) + \det(L_\infty)$.
Then $\sigma(L_0) - \sigma(L_+) = 1$ and $\sigma(L_\infty) - \sigma(L_+) = e$,
where $e := \#\text{neg crossings in } D_\infty - \#\text{neg crossings in } D_+$.

Proof. Take $G_* = G(D_*)$ so that c_0 is (-) and of Type I in D_+ and R_0 is as shown for D_+, D_0, and D_∞:

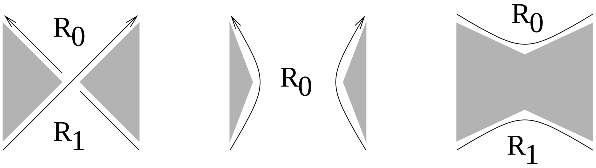


Figure: Coloring convention for c_0

$$G_+ = \begin{pmatrix} a & v \\ v^T & G_0 \end{pmatrix}; \quad G_\infty = \begin{pmatrix} a+1 & v \\ v^T & G_0 \end{pmatrix},$$

where G_0 may be assumed to be diagonal with entries $\alpha_i \neq 0$.

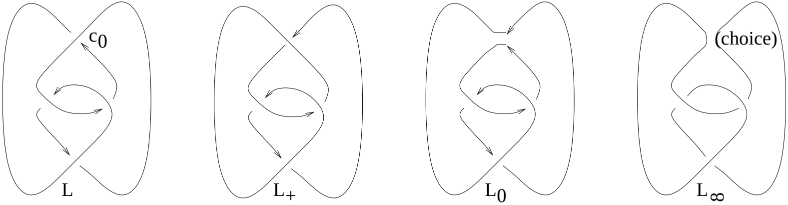
A closer look at the matrices G_*

Let $\widetilde{G}$ be the pre-Goeritz matrix associated with G. Then

$$\widetilde{G}_+ = \left(\begin{array}{cc|c} c & b & v_0 \\ b & a & v \\ \hline v_0^T & v^T & (G_0) \end{array}\right), \quad \widetilde{G}_\infty = \left(\begin{array}{cc|c} c-1 & b+1 & v_0 \\ b+1 & a+1 & v \\ \hline v_0^T & v^T & (G_0) \end{array}\right)$$

$$\text{with } \widetilde{G}_0 = \left(\begin{array}{c|c} a+2b+c & v+v_0 \\ \hline v^T+v_0^T & (G_0) \end{array}\right).$$

Where our example falls short



$$G_+ = \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix}; \quad G_0 = \begin{pmatrix} 2 & -2 \\ -2 & 1 \end{pmatrix}; \quad G_\infty = \begin{pmatrix} 1 \end{pmatrix}.$$

Recall

$$G_+ = \begin{pmatrix} a & v \\ v^T & G_0 \end{pmatrix}; \quad G_\infty = \begin{pmatrix} a+1 & v \\ v^T & G_0 \end{pmatrix}.$$

Then $\det(G_+) = \beta \cdot \det(G_0)$ and $\det(G_\infty) = (\beta + 1) \cdot \det(G_0)$, where

$$\beta = a - \sum_{i=1}^{n} \frac{v_i^2}{\alpha_i}.$$

Establishing $\sigma(L_0) - \sigma(L_+) = 1$

Recall Gordon and Litherland's result Equation (1):

$$\sigma(L) = \text{signature}(G) - \mu(D).$$

$\det(L_+) = \det(L_0) + \det(L_\infty)$ gives $|\beta| = 1 + |\beta + 1|$, so $\beta < -1$.

$$\text{signature}(G_+) = \text{signature}(G_\infty) = \text{signature}(G_0) - 1 \qquad (2)$$

Since c_0 is of Type I, $\mu(D_+) = \mu(D_0)$. Then $\sigma(L_+) = \sigma(L_0) - 1$.

Recall that the resolution D_∞ is *unoriented*. An arbitrary assignment changes some arcs. Changing the direction of an arc at a crossing reverses both the sign and the type.

Establishing $\sigma(L_\infty) - \sigma(L_+) = e$

For fixed incidence number $\mu \in \{\pm 1\}$ and type $t \in \{I, II\}$, denote by $k(\mu, t)$ the number of such crossings in D_+ (excluding c_0) which change type (and sign) in D_∞. Then

$$\mu(D_\infty) - \mu(D_+) = k(+1, I) - k(-1, I) - k(+1, II) + k(-1, II) = -e.$$

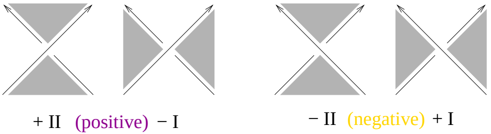


Figure: Positive and negative crossings. □

Long Exact Sequences

Proposition (*Khovanov, Viro, Rasmussen*)
There are long exact sequences

$$\begin{aligned} \ldots \to \quad & \widetilde{Kh}^{i-e-1, j-\frac{3e}{2}-1}(L_\infty) \to \widetilde{Kh}^{i,j}(L_+) \to \widetilde{Kh}^{i, j-\frac{1}{2}}(L_0) \\ \to \quad & \widetilde{Kh}^{i-e, j-\frac{3e}{2}-1}(L_\infty) \to \ldots \end{aligned}$$

and

$$\begin{aligned} \ldots \to \quad & \widetilde{Kh}^{i, j+\frac{1}{2}}(L_0) \to \widetilde{Kh}^{i,j}(L_-) \to \widetilde{Kh}^{i-e+1, j-\frac{3e}{2}+1}(L_\infty) \\ \to \quad & \widetilde{Kh}^{i+1, j+\frac{1}{2}}(L_0) \to \ldots \end{aligned}$$

where e is as in the Lemma above.

Switching to the δ Grading

With $\delta = j - i$, these become

$$\ldots \to \widetilde{Kh}^{*-\frac{e}{2}}(L_\infty) \to \widetilde{Kh}^{*}(L_+) \to \widetilde{Kh}^{*-\frac{1}{2}}(L_0) \to \widetilde{Kh}^{*-\frac{e}{2}-1}(L_\infty) \to \ldots$$

and

$$\ldots \to \widetilde{Kh}^{*+\frac{1}{2}}(L_0) \to \widetilde{Kh}^{*}(L_-) \to \widetilde{Kh}^{*-\frac{e}{2}}(L_\infty) \to \widetilde{Kh}^{*-\frac{1}{2}}(L_0) \to \ldots$$

The Exact Triangle We Want

Corollary (*Manolescu, Ozsváth*) Assume that $\det(L_A)$, $\det(L_B) \neq 0$ and $\det(L) = \det(L_A) + \det(L_B)$ with L_A, L_B its two resolutions at a crossing as above. Then there is an exact triangle:

$$\ldots \to \widetilde{Kh}^{*-\frac{\sigma(L_B)}{2}}(L_B) \to \widetilde{Kh}^{*-\frac{\sigma(L)}{2}}(L) \to \widetilde{Kh}^{*-\frac{\sigma(L_A)}{2}}(L_A)$$
$$\to \widetilde{Kh}^{*-\frac{\sigma(L_B)}{2}}(L_B) \to \ldots$$

Just in case you forgot...

This gives us the result we want:

Main Theorem (*Manolescu, Ozsváth*). Quasi-alternating links are Khovanov homologically thin (over $R = \mathbb{Z}$) on $\delta = \sigma/2$, where σ is the signature of the link.

Thank you!

References

- O. Dasbach, D. Futer, E. Kalfagianni, X.-S. Lin, N. Stoltzfus, *Alternating Sum Formulae for the Determinant and Other Link Invariants*. arXiv:math/0611025 2006.
- C. Manolescu, P. Ozsváth, *On the Khovanov and Knot Floer Homologies of Quasi-Alternating Links*. arXiv:math/0708.3249 2007