HERMITIAN K-THEORY, DERIVED EQUIVALENCES AND KAROUBI'S FUNDAMENTAL THEOREM, `DRAFTGWSCH2.TEX`

MARCO SCHLICHTING

This is a draft which is not (yet) intended for publication!

Abstract. We study higher Grothendieck-Witt groups alias algebraic hermitian K-theory in the framework of exact categories with weak equivalences and duality; generalizing Karoubi's hermitian K-theory of rings. We prove a version of Thomason localization for this theory (when 2 is invertible), extending an exact sequence of Walter to the left. As corollary, we obtain a new, algebraic, and more general proof of Karoubi's fundamental theorem in hermitian K-theory. The localization theorem also implies various (Zariski, Nisnevich) descent theorems for hermitian K and G-theory of (possibly singular) schemes. We also explain the relation of hermitian K-theory to Ranicki's $\mathbb{L}$-groups and Balmer's triangular Witt-grous.

Introduction

Our main result below is a generalization of the K-theory results [Wal85], [TT90], an analog of the Witt-theory result [Bal00], and extends Walter's exact sequence [Wal03] to the left.

Main Theorem. *Let $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be a sequence of complicial functors between complicial exact categories with weak equivalences and duality satisfying 2.1 $(*)$. Assume that the associated sequence of triangulated categories $\mathcal{T}\mathcal{A} \to \mathcal{T}\mathcal{B} \to \mathcal{T}\mathcal{C}$ is exact. Then the sequence of spectra*

$$GW(\mathcal{A}) \to GW(\mathcal{B}) \to GW(\mathcal{C})$$

is a homotopy fibration.

As in algebraic K-theory, the proof of the main theorem relies on an additivity theorem [Wal85, section 1.4], a fibration theorem [Wal85, Theorem 1.6.4] and an approximation theorem [Wal85, Theorem 1.6.7], [TT90, 1.9.8]. The higher Grothendieck-Witt theory version of Additivity and the fibration theorem were proved in [Sch08], don't require condition $(*)$ to hold and work characteristic free. However, our proof of the "invariance under derived equivalence" theorem 2.9 (a version of approximation) needs condition $(*)$.

A consequence of our main theorem is a version of Karoubi's fundamental theorem in hermitian K-theory. A complicial exact category with weak equivalences and duality $(\mathcal{E}, w, \sharp, \eta)$ comes with a family of dualities $\sharp_i$, $i \in \mathbb{Z}$.

Date: Fall 06 - Summer 07.

Theorem. *Let $(\mathcal{E}, w, \sharp, \eta)$ be a complicial exact category with duality satisfying 2.1 $(*)$. Then there is a homotopy fibration of spectra*

$$GW(\mathcal{E}, w, \sharp_n, \eta) \xrightarrow{F} K(\mathcal{E}, w) \xrightarrow{H} GW(\mathcal{E}, w, \sharp_{n+1}, \eta),$$

where F and H are the "forgetful" and "hyperbolic" functors, respectively.

With the easy identifications of Karoubi's U and V-theories as

$${}_{\varepsilon}U(\mathcal{E}) \simeq {}_{\varepsilon}GW(\mathcal{E}, \sharp_{-1}),\ \ {}_{\varepsilon}V(\mathcal{E}) \simeq \Omega\ {}_{\varepsilon}GW(\mathcal{E}, \sharp_1)\ \text{ and }\ {}_{\varepsilon}GW(\mathcal{E}, \sharp_{-1}) \simeq {}_{-\varepsilon}GW(\mathcal{E}, \sharp_1),$$

we obtain a new proof and a generalization of the more familiar version of Karoubi's fundamental theorem

$$\Omega\ {}_{\varepsilon}U(\mathcal{E}) \simeq {}_{-\varepsilon}V(\mathcal{E}).$$

Under the assumption $(*)$, we identify in section 5 the colimit of a sequence of spectra $GW(\sharp) \to \Omega^{-1}GW(\sharp_{-1}) \to \Omega^{-2}GW(\sharp_{-2}) \to ...$ with Ranicki's $\mathbb{L}$-theory spectrum [Ran92] whose homotopy groups $\pi_i\mathbb{L}$ coincide with Balmer's triangular Witt-groups W^{-i}. In particular, our main theorem also holds for $\mathbb{L}$-theory in place of higher Grothendieck-Witt theory – a fact proved some time ago in [Bal00] (though not in the form of spectra) and in [Ran92, ??].

For a complicial exact category with weak equivalences and duality $(\mathcal{E}, w, *, \eta)$, the K-theory spectrum $K(\mathcal{E}, w)$ carries a canonical action of the two element group $\mathbb{Z}/2$. We denote by $K(\mathcal{E}, w)_{h\mathbb{Z}/2}$ the homotopy orbit spectrum. Generalizing a result of Kobal [Kob99], we obtain the following as a consequence of our version of Karoubi's fundamental theorem.

Theorem. *Let $(\mathcal{E}, w, \sharp, \eta)$ be a complicial exact category with weak equivalences and duality satisfying $(*)$ (see 2.1). Then there is a homotopy fibration of spectra*

$$K(\mathcal{E}, w)_{h\mathbb{Z}/2} \longrightarrow GW(\mathcal{E}, w, \sharp, \eta) \longrightarrow \mathbb{L}(\mathcal{E}, w, \sharp, \eta).$$

An analogous statement for the homotopy fixed point spectrum generalizing [Wil05] and [Kob99] is given in **??**.

0.1. **Avertissement.**

0.2. **Terminology.** We say that a map of $(n-1)$-connected spectra is an *equivalence up to π_n* if its cofibre has non-trivial homotopy groups only in degree n. We say that a commutative square of $(n-1)$-connected spectra is *homotopy cartesian up to π_n* if the total cofibre of the square has only non-trivial homotopy groups in degree n. This is equivalent to the map on vertical (or horizontal) cofibres of the square to be an equivalence up to π_n.

Contents

Part 1. Terminology and recollection from [Sch08]

We freely use terminology and results from [Sch08] with the exception that we write $GW_{\geq 0}$ for the (-1)-connected specrtrum whose Ω^{∞}-space was denoted by GW in [Sch08].

1. Complicial exact categories

All known proofs of the analog in algebraic K-theory of our abstract localization theorem 3.5 need more structure than just that of an "exact category with weak equivalences". Usually one demands the existence of a "cylinder functor" ([Wal85], [TT90]), which, in the cases of interest to us, amounts to giving an action of the category $\mathrm{Ch}^b(\mathbb{Z})$ of bounded chain complexes of finitely generated free $\mathbb{Z}$-modules.

In this section, we introduce the hermitian analog: complicial exact categories with weak equivalences and duality. These are the exact categories with weak equivalences and duality which carry an action of $\mathrm{Ch}^b(\mathbb{Z})$ compatible with involutions.

Many categories with duality carry an action by a closed symmetric monoidal category. We start by reviewing the necessary concepts regarding such categories.

1.1. **Closed symmetric monoidal categories.** Recall that a *symmetric monoidal category* is a category $\mathcal{A}$ equipped with a functor $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, an (unit) object $\mathbb{1} \in \mathcal{A}$, and functorial isomorphisms $a : (X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$, $l : \mathbb{1} \otimes X \to X$, $r : X \otimes \mathbb{1} \to X$, and $c : X \otimes Y \cong Y \otimes X$ making certain "coherence" diagrams commute []. In order to simplify formulas, we may assume that a, l, and r are the identity morphisms. This is justified by the fact that any symmetric monoidal category is equivalent to one where a, l, and r are the identity morphisms [**?**].

A *closed symmetric monoidal category* is a symmetric monoidal category $\mathcal{A}$ together with a functor $[\ ,\ ] : \mathcal{A}^{op} \times \mathcal{A} \to \mathcal{A}$ and functorial evaluation $e : [X, Y] \otimes X \to$

Y and coevaluation $d : X \to [Y, X \otimes Y]$ such that the compositions

$$[A,X] \xrightarrow{d} [A,[A,X]\otimes A] \xrightarrow{[1,e]} [A,X] \quad \text{and} \quad X \otimes A \xrightarrow{d\otimes 1} [A, X\otimes A]\otimes A \xrightarrow{e} X \otimes A$$

are the identity maps. A natural transformation which is built out of iterated application of $\otimes$, $[\ ,\ ]$, a, c, d, e, l, r is called admissible. There is a coherence theorem which roughly says, that two admissible natural transformations coincide provided they have the same graph (defined in [KML71]) and no term $[\ \ , \mathbb{1}]$ occurs in the formula. For details, the reader is referred to [KML71].

For every object A of $\mathcal{A}$, we have a functor $\sharp_A : \mathcal{A}^{op} \to \mathcal{A} : X \mapsto [X, A]$. The composition $X \xrightarrow{d} [[X,A], X\otimes[X,A]] \xrightarrow{[1,c]} [[X,A],[X,A]\otimes A] \xrightarrow{[1,e]} [[X,A],A]$ defines a natural transformation

$$\eta^A_X : X \to X^{\sharp_A\sharp_A} = [[X,A],A] \tag{1}$$

which satisfies $(\eta^A_X)^{\sharp_A}\eta^A_{X^{\sharp_A}} = 1_{X^{\sharp_A}}$ [KML71] so that $(\mathcal{A}, \sharp_A, \eta^A)$ is a category with duality. We may write $\sharp$ and η for $\sharp_{\mathbb{1}}$ and $\eta^{\mathbb{1}}$.

There is a unique admissible natural transformation

$$\mathrm{can} : [A,B]\otimes[X,E] \to [A\otimes X, B\otimes E]. \tag{2}$$

It is associative and unital in the obvious sense. Moreover, the diagram

$$\begin{tikzcd} E\otimes X \arrow[r, "\eta^{A\otimes B}_{E\otimes X}"] \arrow[d, "\eta^A_X\otimes\eta^B_X"'] & {[[E\otimes X, A\otimes B], A\otimes B]} \arrow[d, "{[\mathrm{can},1]}"] \\ {[[E,A],A]\otimes[[X,B],B]} \arrow[r, "\mathrm{can}"] & {[[E,A]\otimes[X,B], A\otimes B]} \end{tikzcd} \tag{3}$$

commutes, since, by the coherence theorem, there is a unique admissible natural transformation from the upper left corner to the lower right corner. In summary,

$$(\mathcal{A}, \sharp_A)\times(\mathcal{A},\sharp_B) \xrightarrow{(\otimes,\mathrm{can})} (\mathcal{A}, \sharp_{A\otimes B}) \tag{4}$$

defines a functor of categories with duality which is associative and unital (with unit $(\mathbb{1}, 1) \to (\mathcal{A}, \sharp)$).

1.2. **Remark.** A map $\mu : A\otimes B \to E$ in a closed symmetric monoidal category defines, by adjunction, a map

$$\varphi_\mu : A \xrightarrow{d} [B, A\otimes B] \xrightarrow{[1,\mu]} [B,E] = B^{\sharp_E}$$

which satisfies $\varphi_\mu^{\sharp_E}\circ\eta^E_B = \varphi_{\mu\circ c}$ where $\mu\circ c : B\otimes A \xrightarrow{c} A\otimes B \xrightarrow{\mu} C$. In particular, a map $\mu : A\otimes A \to E$ with $\mu\circ c = \mu$ defines a symmetric form $\varphi_\mu : A \to A^{\sharp_E}$.

1.3. **The category of dg R-modules.** Here is a well-known example of a closed symmetric monoidal category. Let R be a commutative ring with unit, and let dgR-Mod be the category of differential graded R-modules (and maps the closed degree 0 morphisms) [Kel94]. This is the same as the category of complexes of R-modules and chain maps as morphisms. Our convention is that differentials raise the degree (by 1). Usual tensor product $\otimes = \otimes_R$ and internal hom functors $[\ \ ,\ \ ]$ of complexes make dgR-Mod into a closed symmetric monoidal category with unit $\mathbb{1} = R$, the chain complex which is R concentrated in degree 0.

More precisely, tensor product and internal homs are defined by

$$(A \otimes B)^n = \bigoplus_{i+j=n} A^i \otimes B^j$$

with differential $d_{A\otimes B}(x \otimes y) = d_A x \otimes y + (-1)^{|x|} x \otimes d_B y$, and

$$[A,B]^n = \Pi_{i+j=n}[A^{-i}, B^j]$$

with differential $d_{[A,B]}(f) = d_B \circ f - (-1)^{|f|} f \circ d_A$. Note that the differentials are defined in such a way that evaluation $e : [A,B] \otimes A \to B : f \otimes x \mapsto f(x)$, coevaluation $d : A \to [B, A \otimes B] : a \mapsto (b \mapsto a \otimes b)$ and commutativity $c : A \otimes B \to B \otimes A : x \otimes y \mapsto (-1)^{|x||y|} y \otimes x$ are maps of chain-complexes.

The double dual identification $\eta^A_X : X \to X^{\sharp_A \sharp_A} = [[X,A],A] : x \mapsto \eta_X(x)$ of 1.1 (1) is the map satisfying $\eta^A_X(x)(f) = (-1)^{|x||f|} f(x)$ for $f \in X^{\sharp_A}$. The map can : $[X,A] \otimes [Y,B] \to [X \otimes Y, A \otimes B]$ of 1.1 (2) is given by the formula $[\mathrm{can}(f \otimes g)](x \otimes y) = (-1)^{|g||m|} f(x) \otimes g(y)$.

As usual, a sequence $(A,d) \to (B,d) \to (C,d)$ of dg R-modules is exact if $A \to B \to C$ is an exact sequence of (graded) R-modules (forgetting the differentials). A map $(A,d) \to (B,d)$ of dg R-modules is called *weak equivalence* or *quasi-isomorphism* if it induces an isomorphism of cohomology modules.

1.4. **The category $\mathrm{Ch}^b(R)$ of finite, semi-free dg R-modules.** A dg R-module (A,d) is *finite and semi-free* if A is a finitely generated free graded R-module, that is, A has a finite homogeneous R-basis. We denote by $\mathrm{Ch}^b(R) \subset dgR$-Mod the full subcategory of finite and semi-free dg R-modules. This is the same as the category of bounded chain complexes of finitely generated free R-modules. The closed symmetric monoidal structure of dgR-Mod respects finite, semi-free dg R-modules so that it makes $\mathrm{Ch}^b(R)$ into a closed symmetric monoidal category. The category $\mathrm{Ch}^b(R)$ inherits the notion of exact sequences and that of weak equivalences from dgR-Mod.

1.5. **Complicial exact categories.** Fix a commutative ring with unit R, and write $\otimes$ for $\otimes_R$. An exact category $\mathcal{E}$ is called *R-complicial* if it is equipped with a bi-exact tensor product

$$\otimes : \mathrm{Ch}^b(R) \times \mathcal{E} \to \mathcal{E}$$

which defines an associative and unital left action of the monoidal category $\mathrm{Ch}^b(R)$ on $\mathcal{E}$. The words "associative" and "unital" mean that there are natural isomorphisms $A \otimes (B \otimes X) \cong (A \otimes B) \otimes X$ and $\mathbb{1} \otimes X \cong X$ making the usual pentagonal and triangular diagrams [**?**, VII §1] commute (see also [Gra76] for the definition of an action of a monoidal category on another category).

To simplify formulas, we may sometimes suppress the tensor product from the notation, so that, for instance, the associativity isomorphism above will be written as $A(BX) \cong (AB)X$.

1.6. **The fundamental exact sequence.** Let C be the dg-R-algebra freely generated in degree 0 by the unit ε and in degree -1 by an element ν subject to the relations $d\nu = \varepsilon$, $\nu^2 = 0$. Note that C is a commutative dg-R-algebra. We denote by $p : C \to T$ the cokernel of the dg-R-module map $i : \mathbb{1} \to C : 1 \mapsto \varepsilon$. It is the dg-$R$-module T freely generated in degree -1 by the image ν_T of ν. We denote by Γ the *fundamental exact sequence* in $\mathrm{Ch}^b(R)$

$$\Gamma \qquad \mathbb{1} \rightarrowtail^{i} C \overset{p}{\twoheadrightarrow} T. \tag{5}$$

1.7. **The Frobenius structure of a complicial exact category.** Let $\mathcal{E}$ be a complicial exact category, and keep in mind that we agreed to write AX instead of $A \otimes X$ for $A \in \mathrm{Ch}^b$ and $X \in \mathcal{E}$. A map $i : X \to Y$ in $\mathcal{E}$ which has a cokernel in $\mathcal{E}$ is called *Frobenius inflation* if for all maps $f : X \to CU$ in $\mathcal{E}$ there is a map $f' : Y \to CU$ such that $f'i = f$. Dually, a map $p : Y \to Z$ in $\mathcal{E}$ which has a kernel in $\mathcal{E}$ is called *Frobenius deflation* if for all maps $f : CU \to Z$ in $\mathcal{E}$ there is a map $f' : CU \to Y$ such that $pf' = f$.

1.8. **Lemma.** *Let $\mathcal{E}$ be a complicial exact category. Then*

1. *$X \to CX$ is a Frobenius inflation, $C^\sharp X \to X$ is a Frobenius deflation.*
2. *Frobenius inflations (deflations) are inflations (deflations) in $\mathcal{E}$.*
3. *Frobenius inflations (deflations) are closed under composition.*
4. *Frobenius inflations (deflations) are preserved under push-outs (pull-backs)*
5. *Split injections (surjections) are Frobenius inflations (deflations).*

Proof. Recall that C is a commutative dg R-algebra with multiplication $C \otimes C \to C$ and unit map $\varepsilon : \mathbb{1} \to C$.

Given a map $f : X \to CU$, let $f' : CX \to CU$ be the composition $CX \overset{1\otimes f}{\to} CCU \overset{\mu\otimes 1}{\to} CU$. We have $f'i = f$ since the composition $C \overset{1\otimes\varepsilon}{\to} C \otimes C \overset{\mu}{\to} C$ is the identity. Since the map $X \to CX$ has a cokernel, namely TX, it follows that it is a Frobenius inflation. The proof that $C^\sharp X \to X$ is a Frobenius deflation is similar, using the fact that $C^\sharp = [C, \mathbb{1}]$ is a co-algebra.

Let $X \to Y$ be a Frobenius inflation, then there is a map $Y \to CX$ such that the composition $X \to Y \to CX$ is the Frobenius inflation $X \to CX$. Since $X \to CX$ is an inflation in $\mathcal{E}$, and $X \to Y$ has a cokernel, the map $X \to Y$ is an inflation, too. The proof that Frobenius deflations are deflations is dual.

The rest follows easily from the definition of Frobenius inflations and deflations. □

1.9. **Lemma/Definition.** *Let $X \to Y \to Z$ be a conflation in $\mathcal{E}$. Then $X \to Y$ is a Frobenius inflation iff $Y \to Z$ is a Frobenius deflation. Such conflations are called* Frobenius conflations.

Proof. The map $CX \to TX$ is a Frobenius deflation because it is isomorphic to $C^\sharp TX \to TX$ via the isomorphism $C \to [C, T] = C^\sharp T$ which is adjoint to $C \otimes C \overset{\mu}{\to} C \to T$.

Let $X \to Y$ be a Frobenius inflation, then there is a map $Y \to CX$ such that the composition $X \to Y \to CX$ is the canonical Frobenius inflation $X \to CX$. Passing to quotients, we see that $Y \to Z$ is a pull-back of $CX \to TX$. Since the latter is a Frobenius deflation, so is $Y \to Z$.

The other direction is dual. □

Recall that a Frobenius category is an exact category with enough projectives and injectives, and the class of projectives and injectives coincide.

1.10. **Corollary.** *The category $\mathcal{E}$ equipped with the Frobenius conflations is a Frobenius exact category. An object is injective/projective iff it is a direct factor of an object of the form CX.* □

The stable category $\underline{\mathcal{E}}$ of an R-linear Frobenius exact category $\mathcal{E}$ is the category which has the same objects as $\mathcal{E}$, and whose morphisms are the morphisms in $\mathcal{E}$ modulo those which factor through some injective projective object. The stable category $\underline{\mathcal{E}}$ has a canonical structure of an R-linear triangulated category [**?**], [**?**]. In the case of the Frobenius structure associated with an R-complicial exact category $\mathcal{E}$, a sequence in the stable category $\underline{\mathcal{E}}$ is a distinguished triangle iff it is isomorphic in $\underline{\mathcal{E}}$ to a sequence of the form

$$X \xrightarrow{f} Y \to C(f) \to TX$$

where $Y \to C(f)$ is the pushout of $X \to CX$ along f, and $C(f) \to TX$ is the induced map which is zero on Y and on CX it is the quotient map $CX \to TX$.

1.11. **Definition.** An exact category with weak equivalences $(\mathcal{E}, w)$ is called *R-complicial* if $\mathcal{E}$ is R-complicial, and the tensor product preserves weak equivalences in both variables.

In this situation, the fully exact subcategory $\mathcal{E}^w \subset \mathcal{E}$ of w-acyclic objects (that is, of those objects X for which $0 \to X$ is a weak equivalence) is an R-complicial exact subcategory of $\mathcal{E}$. In particular, $(\mathcal{E}, \mathcal{E}^w)$ is a Frobenius pair in the sense of [Sch04]. The associated triangulated or derived category $\mathcal{T}(\mathcal{E}, w)$ of $(\mathcal{E}, w)$ is the category $w^{-1}\mathcal{E}$, obtained from $\mathcal{E}$ by formally inverting the weak equivalences in $\mathcal{E}$. It is equivalent to the Verdier quotient $\underline{\mathcal{E}}/\underline{\mathcal{E}^w}$ of the stable category of $\mathcal{E}$ modulo the stable category of $\mathcal{E}^w$, and thus carries a canonical structure of an R-linear triangulated category.

Now we add dualities to the picture.

1.12. **Categories with dualities in a closed symmetric monoidal category.** Let $\mathcal{A}$ be a closed symmetric monoidal category. A *category with dualities in $\mathcal{A}$* is a category $\mathcal{U}$ together with functors

$$\mathcal{A} \times \mathcal{U} \xrightarrow{\otimes} \mathcal{U} : (A, X) \mapsto A \otimes X \qquad \text{and}$$

$$\mathcal{U}^{op} \times \mathcal{A} \xrightarrow{[\ ,\ ]} \mathcal{U} : (X, A) \mapsto [X, A]$$

and natural transformations η and can as in 1.1 (1) and 1.1 (2) with $X \in \mathcal{U}$ and $A, B, E \in \mathcal{A}$. As usual, we write $\sharp_A$ for the functor $\mathcal{U}^{op} \to \mathcal{U} : X \mapsto [X, A]$. We require that $\otimes$ and can are associative and unital (from the left), η satisfies $(\eta^A_X)^{\sharp_A} \circ \eta^A_{X^{\sharp_A}} = 1_{X^{\sharp_A}}$ and 1.1 (3) commutes where X is in $\mathcal{U}$ and A, B, E are in $\mathcal{A}$. In particular,

$$(\mathcal{A}, \sharp_A) \times (\mathcal{U}, \sharp_B) \xrightarrow{(\otimes, \mathrm{can})} (\mathcal{U}, \sharp_{A \otimes B}) \tag{6}$$

is a functor of categories with duality which defines an associative and unital left action.

Of course, any closed symmetric monoidal category $\mathcal{A}$ defines a category with dualities in $\mathcal{A}$. Also, if $\mathcal{A} \to \mathcal{B}$ is a closed symmetric monoidal functor, then $\mathcal{B}$ is a category with dualities in $\mathcal{A}$ via $\mathcal{A} \to \mathcal{B}$.

1.13. **Example.** *Complexes of O_X-modules.* Let X be a scheme over $\operatorname{Spec} R$. The category $\mathrm{Ch}(O_X\text{-Mod})$ of complexes of O_X-modules is closed symmetric monoidal via the same formulas as in 1.3. The map $R \to \Gamma(X, O_X)$ defines a closed symmetric monoidal functor $\otimes_R O_X : \mathrm{Ch}^b(R) \to \mathrm{Ch}(O_X\text{-Mod})$. Let K be any complex of O_X-modules, then, varying $X \in \mathrm{Ch}(O_X\text{-Mod})$ and $A, B, E \in \mathrm{Ch}^b(R)$, the data $A \otimes_R X$, $[X, A]_K = [X, A \otimes K]$, $\eta^{A \otimes K}$ and $\mathrm{can} : [A, B] \otimes [X, E]_K \to [A \otimes X, B \otimes E]_K$ define on $\mathrm{Ch}(O_X\text{-Mod})$ the structure $(\otimes, [\ \ ,\ \ ]_K, \eta^K, \mathrm{can})$ of a category with dualities in $\mathrm{Ch}^b(R)$.

As usual, a sequence $(A, d) \to (B, d) \to (C, d)$ of complexes of O_X-modules is exact if $A \to B \to C$ is an exact sequence of O_X-modules (forgetting the differentials), and a map $(A, d) \to (B, d)$ of chain complexes of O_X-modules is a *weak equivalence* or *quasi-isomorphism* if it induces an isomorphism of cohomology sheaves.

We will come back to this example in part 3.

1.14. **Definition.** Let R be a commutative ring with unit. An *R-complicial exact category with weak equivalences and duality* is an exact category with weak equivalences $(\mathcal{A}, w)$ together with a structure $(\otimes, [\ \ ,\ \ ], \eta, \mathrm{can})$ on $\mathcal{A}$ of a category with dualities in the symmetric monoidal category $\mathrm{Ch}^b(R)$ of finite, semi-free dg R-modules such that $\eta^{\mathbb{1}}$ is a natural weak equivalence and such that the functors $\otimes : \mathrm{Ch}^b(R) \times \mathcal{A} \to \mathcal{A}$ and $[\ \ ,\ \ ] : \mathcal{A}^{op} \times \mathrm{Ch}^b(R) \to \mathcal{A}$ are bi-exact and preserve weak equivalences.

Its *associated triangulated* or *derived category* $\mathcal{TA}$ is the category $w^{-1}\mathcal{A}$, obtained from $\mathcal{A}$ by formally inverting weak equivalences. Since $(\mathcal{A}, w, \otimes)$ is an R-complicial exact category with weak equivalences, $\mathcal{TA}$ has a canonical structure of a R-linear triangulated category, see definition 1.11.

A *complicial exact category with weak equivalences and duality* is simply a $\mathbb{Z}$-complicial exact category with weak equivalences and duality.

1.15. **Remark.** If $(\mathcal{A}, w, \otimes, [\ \ ,\ \ ], \eta, \mathrm{can})$ is an R-complicial exact category with duality. Then changing η into $-\eta$ defines another R-complicial exact category with duality $(\mathcal{A}, w, \otimes, [\ \ ,\ \ ], -\eta, \mathrm{can})$.

1.16. **Remark.** Definition 1.14 implies that $\mathrm{can} : [A, B] \otimes [X, E] \to [A \otimes X, B \otimes E]$ is an isomorphism for $A, B, E \in \mathrm{Ch}^b(R)$, $X \in \mathcal{A}$. We can see this as follows. Using biexactness of $\otimes$ and $[\ \ ,\ \ ]$ one reduces the claim to showing that can is an isomorphism when $A = R[i]$ and $B = R[j]$. The fact that can is an isomorphism in this case follows from the requirement that it be associative and unital.

1.17. **Invertible objects.** Recall that an object L in a closed symmetric monoidal category is called *invertible* if the evaluation map $e : [L, \mathbb{1}] \otimes L \to \mathbb{1}$ is an isomorphism. For an invertible object, the map $\mathrm{can} : [B, A] \otimes L = [B, A] \otimes [\mathbb{1}, L] \to [B, A \otimes L]$ is an isomorphism. The signature sgn_L of L is the composition of the isomorphisms $\mathrm{sgn}_L : \mathbb{1} \xrightarrow{d} [L, L] \xleftarrow{\mathrm{can}} L^\sharp \otimes L \xrightarrow{e} \mathbb{1}$.

If $e_1, ..., e_n$ is a homogeneous base of an object A of $\mathrm{Ch}^b(R)$, we denote by $e_1^\sharp, ..., e_n^\sharp$ its dual base, the base of $A^\sharp = [A, \mathbb{1}]$ defined by the requirements $e_i^\sharp(e_i) = (-1)^{|e_i|}$ and $e_i^\sharp(e_j) = 0$ for $i \neq j$.

Let L be a dg R-module freely generated by an element ν of degree $|\nu|$. Since evaluation $e : L^\sharp \otimes L \to \mathbb{1}$ sends the generator $\nu^\sharp \otimes \nu$ to $\nu^\sharp(\nu) = (-1)^{|e_i|}$, the object

L is invertible. Moreover, the sequence of maps $\mathrm{sgn}_L : \mathbb{1} \xrightarrow{d} [L,L] \xleftarrow{\mathrm{can}} L^\sharp \otimes L \xrightarrow{e} \mathbb{1}$ is $1 \mapsto id \mapsto \nu^\sharp \otimes \nu \mapsto (-1)^\nu$, so that the signature of L is $(-1)^{|\nu|}$.

1.18. **Remark.** The requirement in definition 1.14 that $\eta^{\mathbb{1}}$ be a weak equivalence implies that for any invertible object $L \in \mathrm{Ch}^b(R)$, the natural transformation $\eta^L : X \to [[X,L],L]$ is a weak equivalence. This follows from diagram 1.1 (3) with $A = L$, $B = E = \mathbb{1}$. and the fact that can is an isomorphism (1.16). The same argument shows that if $\eta^{\mathbb{1}}$ is an isomorphism then so is η^L for any invertible object L of $\mathrm{Ch}^b(R)$.

1.19. **Remark.** Let $\mathcal{A}$ be an R-complicial exact category, and A, B invertible objects in $\mathrm{Ch}^b(R)$. Let's write as usual $\sharp_A$ for the functor $[\ \ , A]$. Let ε and δ be 1 or -1. Then we have a bi-exact functor

$$(\mathrm{Ch}^b R, \mathrm{quis}, \sharp_A, \varepsilon\eta^A) \times (\mathcal{A}, w, \sharp_B, \delta\eta^B) \overset{(\otimes,\mathrm{can})}{\longrightarrow} (\mathcal{A}, w, \sharp_{A\otimes B}, \varepsilon\delta\eta^{A\otimes B}) \tag{7}$$

of categories with weak equivalences and duality. Moreover, the tensor product $(\otimes, \mathrm{can})$ is associative and unital.

1.20. **Remark.** Let A and B be invertible objects of $\mathrm{Ch}^b(R)$. Any symmetric space (E,φ) in the (exact) category with duality $(\mathrm{Ch}^b R, \sharp_A, \varepsilon\eta^A)$ defines a functor $(E,\varphi) : \mathrm{pt} \to (\mathrm{Ch}^b R, \sharp_A, \varepsilon\eta^A) : \mathrm{pt} \mapsto E$ of categories with (strong) duality, where pt denotes the category with duality which has one object pt and one morphism 1_{pt}. Composition with (7) defines an exact functor

$$(E,\varphi) : (\mathcal{A}, w, \sharp_B, \delta\eta^B) \to (\mathcal{A}, w, \sharp_{A\otimes B}, \varepsilon\delta\eta^{A\otimes B})$$

of categories with weak equivalences and duality which is "tensor product with the symmetric space (E,φ)". Composition of such functors corresponds to tensor product of symmetric spaces.

Now, let L be an invertible object of $\mathrm{Ch}^b(R)$. As in 1.2, the identity map $id_{L^2} : L \otimes L \to L^2$ defines, by adjunction, a map $\rho_L = \varphi_{id_{L^2}} : L \to [L, L^2] = L^{\sharp_{L^2}}$ which is easily seen to be an isomorphism. Since $id_{L^2} \circ c = \mathrm{sgn}_L\, id_{L^2}$, the map satisfies $\rho_L^{\sharp_{L^2}}\eta = \mathrm{sgn}_L\, \rho_L$. In other words, the pair (L, ρ_L) defines a symmetric space in $(\mathrm{Ch}^b(R), \sharp_{L^2}, \mathrm{sgn}_L\, \eta^{L^2})$.

1.21. **Lemma.** *Let $\mathcal{A}$ be an R-complicial exact category with weak equivalences and duality, and let L, M be invertible objects in $\mathrm{Ch}^b(R)$. Then tensor product with the symmetric space (L,ρ_L) defined in 1.20 defines an equivalence of exact categories with weak equivalences and duality*

$$(L,\rho_L) : (\mathcal{A}, w, \sharp_M, \eta^M) \to (\mathcal{A}, w, \sharp_{L^2\otimes M}, \mathrm{sgn}_L\, \eta^{L^2\otimes M}).$$

Proof. The map $\lambda : (L^\sharp)^2 \otimes L^2 \overset{\mathrm{can}\otimes 1}{\longrightarrow} (L^2)^\sharp \otimes L^2 \xrightarrow{e} \mathbb{1}$ defines an isomorphism $(id,\lambda) : (\sharp_{(L^\sharp)^2\otimes L^2 \otimes M}, \eta) \to (\sharp_M, \eta)$ of exact categories with weak equivalences and duality. One checks that the composition $(id,\lambda)\circ(L^\sharp, \rho_{L^\sharp}) \circ (L,\rho_L)$ is naturally isometric to the identity functor of $(\sharp_M,\eta)$. Similarly, there is an isomorphism $(id,\bar\lambda)$ of categories with duality such that $(id,\bar\lambda)\circ(L,\rho)\circ(L^\sharp,\rho)$ is naturally isometric to the identity functor. This implies the claim. $\square$

1.22. **Definition.** A *(complicial) functor of R-complicial exact categories with duality* from $\mathcal{A}$ to $\mathcal{B}$, is a triple (F,φ,ρ) with $F:\mathcal{A}\to\mathcal{B}$ an exact functor preserving weak equivalences, $\rho : A\otimes F(X)\to F(A\otimes X)$ a natural transformation, and $\varphi : F[X,A]\to[FX,A]$ a natural weak equivalence. We require that ρ be associative and unital from the left with respect to $\otimes$ (this implies that ρ is a natural isomorphism as in 1.16), and that the following two diagrams commute

$$\begin{tikzcd}
{[A,B]\otimes F[X,E]} \arrow[r, "\rho", "\cong"'] \arrow[d, "1\otimes\varphi"'] & F([A,B]\otimes[X,E]) \arrow[r, "F(\mathrm{can})", "\cong"'] & F[A\otimes X, B\otimes E] \arrow[d, "\varphi"] \\
{[A,B]\otimes[FX,E]} \arrow[r, "\mathrm{can}", "\cong"'] & {[A\otimes FX, B\otimes E]} & {[F(A\otimes X), B\otimes E]} \arrow[l, "{[\rho,1]}"', "\cong"]
\end{tikzcd} \tag{8}$$

$$\begin{tikzcd}
FX \arrow[r, "F\eta^{\mathcal{A}}"] \arrow[d, "\eta^{\mathcal{B}}"] & F[[X,A],A] \arrow[d, "\varphi"] \\
{[[FX,A],A]} \arrow[r, "{[\varphi,1]}"] & {[F[X,A],A]}
\end{tikzcd} \tag{9}$$

Composition of such functors is composition of the F's, ρ's and φ's, respectively,

1.23. **An equivalent definition.** By remark 1.19 with $A=B=\mathbb{1}$ and $\varepsilon=\delta=1$, an R-complicial exact category with weak equivalences and duality $(\mathcal{A},w,\otimes,[\ ,\ ],\eta,\mathrm{can})$ induces an associative and unital biexact left action

$$(\mathrm{Ch}^b(R),\mathrm{quis},\sharp,\eta)\times(\mathcal{A},w,\sharp,\eta)\overset{(\otimes,\mathrm{can})}{\to}(\mathcal{A},w,\sharp,\eta), \tag{10}$$

where $\sharp=[\ ,\mathbb{1}]$ and $\eta=\eta^{\mathbb{1}}$.

In fact, a left action of $(\mathrm{Ch}^b(R),\mathrm{quis},\sharp,\eta)$ on $(\mathcal{A},w,\sharp,\eta)$ already defines an R-complicial exact category $(\mathcal{A},\otimes,[\ ,\ ]',\eta',\mathrm{can}')$ which is isomorphic to $(\mathcal{A},\otimes,[\ ,\ ],\eta,\mathrm{can})$. Given such a left action (with $\mathrm{can}:A^\sharp\otimes X^\sharp\to(A\otimes X)^\sharp$ a natural isomorphism, and $\eta:X\to X^{\sharp\sharp}$ a natural weak equivalence, $A\in\mathrm{Ch}^b(R)$ and $X\in\mathcal{A}$), we define $[X,A]'=A\otimes X^\sharp$, and η' and can' as the compositions

$$X=\mathbb{1}\otimes X\overset{d\otimes\eta}{\to}[A,A]\otimes X^{\sharp\sharp}\overset{\mathrm{can}\otimes 1}{\leftarrow}A\otimes A^\sharp\otimes X^{\sharp\sharp}\overset{1\otimes\mathrm{can}}{\to}A\otimes(A\otimes X^\sharp)^\sharp=[[X,A]',A]',$$

$$[A,B][X,E]'=[A,B]EX^\sharp\overset{\mathrm{can}\,1}{\leftarrow}BA^\sharp EX^\sharp\overset{1c1}{\to}BEA^\sharp X^\sharp\overset{1\,\mathrm{can}}{\to}BE(AX)^\sharp=[AX,BE]',$$

where on the second line we suppressed the tensor product $\otimes$ in the notation. We leave it as an exercise for the reader to check that with these definitions, $(\mathcal{A},\otimes,[\ ,\ ]',\eta',\mathrm{can}')$ is an R-complicial exact category with duality, and the functor $(id,id,\mathrm{can}):(\mathcal{A},\otimes,[\ ,\ ]',\eta',\mathrm{can}')\to(\mathcal{A},\otimes,[\ ,\ ],\eta,\mathrm{can})$ is an isomorphism of complicial exact categories with duality.

Part 2. **Higher Grothendieck-Witt groups and derived categories**

2. Karoubi-periodicity and invariance under derived equivalences

Let $(\mathcal{A},w,\otimes,[\ ,\ ],\eta,\mathrm{can})$ be a complicial exact category with duality. We write $\sharp_n$ and η^n for $[\ ,T^n]$ and η^{T^n}. Then $(\mathcal{A},w,\sharp_n,\eta^n)$ is an exact category with weak equivalences and duality carrying a left action by $(\mathrm{Ch}^b\,\mathbb{Z},\mathrm{quis},\sharp,\eta)$. Note that by lemma 1.21 (with $L=T$, and noting that $\operatorname{sgn}T=-1$), the two categories with weak equivalences and duality $(\mathcal{A},w,\sharp_n,\eta^n)$ and $(\mathcal{A},w,\sharp_{n+2},-\eta^{n+2})$ are equivalent, so that $GW_{\geq 0}(\mathcal{A}, w, \sharp_n, \eta^n) \cong GW_{\geq 0}(\mathcal{A}, w, \sharp_{n+2}, -\eta^{n+2})$ only depends on n modulo 4.

2.1. **Standing Assumption.** Let $(\mathcal{E}, *)$ be a complicial exact category with weak equivalences and duality. We say that $\mathcal{E}$ satisfies the condition $(*)$ if

- $(*)$ there is a natural transformation $\lambda : id_{\mathcal{E}} \to id_{\mathcal{E}}$ such that
  - (1) for every object X of $\mathcal{E}$ we have $\lambda_{X^*} + {\lambda_X}^* = 1_{X^*}$, and
  - (2) for $A \in \mathrm{Ch}^b\,\mathbb{Z}$, $X \in \mathcal{E}$ we have $\lambda_{A\otimes X} = 1_A \otimes \lambda_X$.

Recall that $\mathcal{E}$ is equipped with a family $\sharp_L$ of dualities with $* = \sharp_{\mathbb{1}}$, where L ranges over the invertible objects of $\mathrm{Ch}^b\,\mathbb{Z}$. Since $X^{\sharp_L} = [X, L]$ is naturally isomorphic to $L \otimes X^*$, we have $\lambda_{X^{\sharp_L}} + (\lambda_X)^{\sharp_L} = \lambda_{L\otimes X^*} + L \otimes \lambda_X^* = 1_L \otimes (\lambda_{X^*} + \lambda_X^*) = 1_{X^{\sharp_L}}$. In other words, $(\mathcal{E}, w, \sharp_L)$ also satisfies $(*)$.

2.2. **Examples.** Here are some examples of complicial categories satisfying $(*)$. If $\mathcal{E}$ is a $\mathbb{Z}[\frac{1}{2}]$-linear category (in which case we say "2 is invertible in $\mathcal{E}$") then we can take as λ the multiplication by $\frac{1}{2}$, that is, a $\mathbb{Z}[\frac{1}{2}]$-complicial exact category with weak equivalences and duality always satisfies $(*)$. If $\mathcal{E}$ is the hyperbolic category $H\mathcal{A}$ of a complicial exact category with weak equivalences $\mathcal{A}$, then we can take $\lambda = (1, 0)$ the natural transformation $(1_X, 0) : (X, Y) \to (X, Y)$. So hyperbolic categories always satisfy $(*)$. If $\mathcal{A}$ is an exact category with duality for which there is a natural transformation $\lambda : id \to id$ satisfying (1) above, then the category $\mathcal{E} = \mathrm{Ch}^b(\mathcal{A})$ satisfies $(*)$. For instance, we can take as $\mathcal{A}$ the category of finitely generated projective modules over the hermitian ring $R[x]$, where R is a commutative ring with trivial involution, and $\bar{x} = 1 - x$. Here λ is multiplication with x.

2.3. **Lemma.** *Let $(\mathcal{A}, w, \eta, \sharp)$ be a complicial exact category with weak equivalences and duality satisfying $(*)$. Then the functor $(w\mathcal{A})_h \to (i\mathcal{T}\mathcal{A})_h : (X, \varphi) \mapsto (X, \varphi)$ induces an isomorphism of abelian monoids of connected components*

$$\pi_0\ (w\mathcal{A})_h \to \pi_0\ (i\mathcal{T}\mathcal{A})_h.$$

Proof. We will show that the functor (11) below is full and essentially surjective

$$w^{-1}(w\mathcal{A})_h \to (i\mathcal{T}\mathcal{A})_h. \tag{11}$$

This clearly implies the claim.

The functor (11) is essentially surjective on objects by the following argument. Let (A, α) be a symmetric space in $w^{-1}\mathcal{A}$. We can write α as a fraction $\alpha = a \circ s^{-1}$ with $a : B \to A^\sharp$ and $s : B \to A \in w$ weak equivalences in $\mathcal{A}$. Let $\varphi = \lambda s^\sharp a + \lambda^\sharp a^\sharp \eta s : B \to B^\sharp$. Then (B, φ) is a symmetric weak equivalence in $\mathcal{A}$, and $s : B \to A$ defines an isometry in $\mathcal{T}\mathcal{A}$ between (B, φ) and (A, α).

To finish, we show that the functor (11) is full. Let (A, α) and (B, β) be objects of $(w\mathcal{A})_h$, and let $[fs^{-1}]$ be an isometry in $\mathcal{T}\mathcal{A}$ from (A, α) to (B, β). In particular, f and s are weak equivalences in $\mathcal{A}$. Using a calculus of fractions in $\mathcal{T}\mathcal{A}$, we can assume the fraction fs^{-1} with $f : E \to B$ and $s : E \to A$, is such that $\alpha_0 = s^\sharp \alpha s$ and $\alpha_1 = f^\sharp \beta f$ are homotopic. Therefore, the difference $\alpha_0 - \alpha_1$ factors as $E \xrightarrow{\varepsilon} I \xrightarrow{\delta} E^\sharp$ where I is w-acyclic. Then the pair $(E \oplus I, \varphi)$ with

$$\varphi = \begin{pmatrix} \alpha_1 & \lambda\delta \\ \lambda^\sharp \delta^\sharp \eta & 0 \end{pmatrix} : E \oplus I \to E^\sharp \oplus I^\sharp$$

defines an object of $(w\mathcal{A})_h$. The functor (11) sends the map

$$(A,\alpha) \xleftarrow{s} (E,\alpha_0) \xrightarrow{\binom{1}{\varepsilon}} (E\oplus I,\varphi) \xleftarrow{\binom{1}{0}} (E,\alpha_1) \xrightarrow{f} (B,\beta)$$

in $w^{-1}(w\mathcal{A})_h$ to fs^{-1} in $(i\mathcal{T}\mathcal{A})_h$. $\square$

2.4. **Lemma.** *Let $(F,\varphi) : (\mathcal{A},w) \to (\mathcal{B},w)$ be a map of exact categories with weak equivalences and duality (φ being a symmetric weak equivalence). Suppose that there are a functor $G : (\mathcal{B},w) {\to} (\mathcal{A},w)$ of exact categories with weak equivalences (no compatibility with dualities required) and zig-zags of natural weak equivalences of functors between FG and $id_{\mathcal{B}}$ and between GF and $id_{\mathcal{A}}$. If $\mathcal{A}$ and $\mathcal{B}$ are complicial and satisfy $(*)$, then (F,φ) induces an equivalence of (-1)-connected Grothendieck-Witt theory spectra*

$$GW_{\geq 0}(\mathcal{A},w) \xrightarrow{\sim} GW_{\geq 0}(\mathcal{B},w).$$

Proof. To simplify notation, we will write throughout this proof $\mathrm{Fun}(\mathcal{A},\mathcal{B})$ for the exact category with weak equivalences and duality $\mathrm{ExFun}(\mathcal{A},w;\mathcal{B},w)$. This is a complicial exact category with weak equivalences and duality satisfifying $(*)$ by the following definitions. One uses the functoriality of $\otimes$, $[\ \ ,\ \ ]$ and λ in $\mathcal{A}$ to define $(A\otimes F)$, $[F,A]$ and $\lambda_F : F \to F$ as $(A\otimes F)(X) = A\otimes F(X)$, $[F,A](X) = [F(X),A]$ and $\lambda_F(X) = \lambda_{F(X)} : F(X) \to F(X)$ for $F \in \mathrm{Fun}(\mathcal{E},\mathcal{A})$. In particular, $\mathrm{Fun}(\mathcal{B},\mathcal{A})$ and $\mathrm{Fun}(\mathcal{A},\mathcal{B})$ are complicial exact categories with weak equivalences and duality satisfying $(*)$.

The hypothesis of the lemma imply that $F : \mathrm{Fun}(\mathcal{B},\mathcal{A}) \to \mathrm{Fun}(\mathcal{B},\mathcal{B}) : H \mapsto F\circ H$ induces an equivalence $\mathcal{T}\,\mathrm{Fun}(\mathcal{B},\mathcal{A}) \to \mathcal{T}\,\mathrm{Fun}(\mathcal{B},\mathcal{B})$ of derived categories with inverse G. By lemma 2.3, (F,φ) induces an isomorphism of abelian monoids

$$\pi_0\ (w\,\mathrm{Fun}(\mathcal{B},\mathcal{A}))_h \xrightarrow{\cong} \pi_0\ (w\,\mathrm{Fun}(\mathcal{B},\mathcal{B}))_h.$$

So there is an object $(H,\psi) \in (w\,\mathrm{Fun}(\mathcal{B},\mathcal{A}))_h$ which under this isomorphism goes to $id_{\mathcal{B}} \in (w\,\mathrm{Fun}(\mathcal{B},\mathcal{B}))_h$. This means that there is a zigzag of natural weak equivalences between $(F,\varphi)\circ(H,\psi)$ and $id_{\mathcal{B}}$ compatible with forms. By lemma **??**, the functors $(F,\varphi)\circ(H,\psi)$ and $id_{\mathcal{B}}$ induce homotopic maps on (-1)-connected hermitian K-theory spectra. In particular, (F,φ) is surjective, and (H,ψ) is injective as maps on higher Grothendieck-Witt groups GW_i, $i \geq 0$.

By construction of H and the existence of G, there are zig-zags of weak equivalences of functors between HF and $id_{\mathcal{A}}$ and between FH and $id_{\mathcal{B}}$. By the argument above (with (H,ψ) in place of (F,φ)), we see that (H,ψ) is also surjective and hence bijective on higher Grothendieck-Witt groups. From the previous paragraph it follows that (F,φ) induces isomorphisms $GW_i(\mathcal{A},w) \to GW_i(\mathcal{B},w)$, $i \geq 0$.

$\square$

2.5. **The mapping cone.** Let R be a commutative ring with unit. Recall that the cone $C \in \mathrm{Ch}^b(R)$ is a commutative dg-R algebra with multiplication $\mu : C\otimes C \to C$ defined in 1.6. As in 1.2, the composition $p\mu : C\otimes C \to T$ defines, by adjunction, a form $\gamma = \varphi_{p\mu} : C \to C^{\sharp_T} = [C,T]$ which is symmetric, that is, $\gamma^{\sharp_T}\eta^T = \gamma$ since $\mu\circ c = \mu$ (see remark 1.2). We obtain in this way a symmetric isomorphism of exact sequences

$$\begin{tikzcd}
\Gamma : \arrow[d, "\gamma"] & \mathbb{1} \arrow[r, "i"] \arrow[d, "d"] & C \arrow[r, "p"] \arrow[d, "\gamma"] & T \arrow[d, "id"] \\
{[\Gamma, T] :} & {[T,T]} \arrow[r, "{[p,1]}"] & {[C,T]} \arrow[r, "{[i,1]}"] & {[\mathbb{1},T]}
\end{tikzcd}$$

from the fundamental exact sequence of 1.6 to its T-dual. In other words, the pair (Γ, γ) defines a symmetric space in the category $(S_2\,\mathrm{Ch}^b(R), \sharp_T, \eta^T)$ of exact sequences of chain complexes with shifted duality $\sharp_T$.

For an exact category with weak equivalences and duality $(\mathcal{A}, w, \sharp, \eta)$, we define a functor

$$(\Delta, \delta) : (S_2 Mor\mathcal{A}, \sharp) \to (\mathcal{A}, \sharp)$$

of exact categories with weak equivalences and duality as follows. Any map $f_* : A_* \to B_*$ of exact sequences can be factored into a sequence of maps $\bar{\Delta}(f)$ of exact sequences as follows

$$\begin{tikzcd}
 & 0 \arrow[r] & A_0 \arrow[r] \arrow[d, "f_0"] \arrow[dr, phantom, "\square"] & A_1 \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d, equal] & 0 & A_* \arrow[dd, "f_*"] \\
\bar{\Delta}(f) & 0 \arrow[r] & B_0 \arrow[r] \arrow[d, equal] & \Delta(f_*) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & A_2 \arrow[r] \arrow[d, "f_2"] & 0 & \\
 & 0 \arrow[r] & B_0 \arrow[r] & B_1 \arrow[r] & B_2 \arrow[r] & 0 & B_*,
\end{tikzcd}$$

where the upper left and lower right squares are bicartesian. This factorization is unique up to unique isomorphism, and therefore defines a functor $\bar{\Delta}$ from $S_2 Mor\mathcal{A}$ to such diagrams. In particular, we have a functor $\Delta : S_2 Mor\mathcal{A} \to \mathcal{A}$ which is $\bar{\Delta}$ composed with evaluation at the middle spot. The duality compatibility isomorphism $\bar{\Delta}(f^\sharp) \to \bar{\Delta}(f)^\sharp$ is the unique map of diagrams inducing the identity of $f_*^\sharp$ on the composition. Its restriction to the middle spot of the diagram defines the duality compatibility isomorphism $\delta : \Delta(f^\sharp) \to \Delta(f)^\sharp$.

We define the cone functor Cone : $(Mor\mathcal{A}, w, \sharp_n, \eta^n) \to (\mathcal{A}, w, \sharp_{n+1}, \eta^{n+1})$ as the composition

$$\text{Cone} : (Mor\mathcal{A}, \sharp_n) \overset{(\Gamma,\gamma)\otimes id}{\longrightarrow} (S_2 Mor\mathcal{A}, \sharp_{n+1}) \overset{(\Delta,\delta)}{\longrightarrow} (\mathcal{A}, \sharp_{n+1}). \tag{12}$$

For any exact category with duality and weak equivalences $\mathcal{A}$ we have a functor $(\mathcal{A}, \sharp_n) \to (Mor\mathcal{A}, \sharp_n) : A \mapsto id_A$ with duality compatibility isomorphism $id : id_{A^{\sharp_n}} \overset{=}{\longrightarrow} id_A^{\sharp_n}$.

2.6. Theorem. *Let $(\mathcal{A}, w)$ be a complicial exact category with weak equivalences and duality satisfying $(*)$. Then the commutative square*

$$\begin{tikzcd}
(\mathcal{A}, w, \sharp_n, \eta^n) \arrow[r, "A \mapsto id_A"] \arrow[d] & (Mor\mathcal{A}, w, \sharp_n, \eta^n) \arrow[d, "\text{Cone}"] \\
(\mathcal{A}^w, w, \sharp_{n+1}, \eta^{n+1}) \arrow[r, hook] & (\mathcal{A}, w, \sharp_{n+1}, \eta^{n+1})
\end{tikzcd} \tag{13}$$

induces a homotopy cartesian square up to π_0 *of* $GW_{\geq 0}$ *spectra and the lower left corner's hermitian* K*-theory is contractible. In particular, there is a homotopy fibration up to* π_0

$$GW_{\geq 0}(\mathcal{A},w,\sharp_n) \longrightarrow GW_{\geq 0}(Mor\mathcal{A},w,\sharp_n) \longrightarrow GW_{\geq 0}(\mathcal{A},w,\sharp_{n+1}).$$

Proof. As usual, the lower left corner's hermitian K-theory is contractible by lemma **??** in view of the natural weak equivalence $0 \to X$ for $X \in \mathcal{A}^w$.

Let v be the set of maps f in $\operatorname{Mor}\mathcal{A}$ for which $\operatorname{Cone}(f)$ is a weak equivalence in $\mathcal{A}$. Then

$$((\operatorname{Mor}\mathcal{A})^v,w,\sharp_n) \to (\operatorname{Mor}\mathcal{A},w,\sharp_n) \to (\operatorname{Mor}\mathcal{A},v,\sharp_n)$$

induces a homotopy fibration up to π_0 of $GW_{\geq 0}$ spectra by theorem **??**. The functor $(\mathcal{A},w,\sharp_n) \to ((\operatorname{Mor}\mathcal{A})^v,w,\sharp_n) = (\operatorname{Mor}_w\mathcal{A},w,\sharp_n) : A \mapsto id_A$ induces an equivalence of $GW_{\geq 0}$ spectra by lemma **??**.

The induced functor $(F,\varphi) = \operatorname{Cone} : (\operatorname{Mor}\mathcal{A},v,\sharp_n) \to (\mathcal{A},w,\sharp_{n+1})$ induces an equivalence of $GW_{\geq 0}$ spectra by lemma 2.4, where $G : (\mathcal{A},w) \to (\operatorname{Mor}\mathcal{A},v)$ sends X to the object $0 \to X$ of $\operatorname{Mor}\mathcal{A}$. We have $FG = id_{\mathcal{A}}$ and a zigzag of natural weak equivalences $id \to H \leftarrow GF$ where H sends an object $f : X \to Y$ of $\operatorname{Mor}\mathcal{A}$ to the object $\operatorname{Cone}(id_X) \to \operatorname{Cone}(f)$, and the natural weak equivalences are given by the natural maps from $f : X \to Y$ to $\operatorname{Cone}(id_X) \to \operatorname{Cone}(f)$ and from $0 \to \operatorname{Cone}(f)$ to $\operatorname{Cone}(id_X) \to \operatorname{Cone}(f)$. □

2.7. **Corollary** (Fundamental theorem). *Let* $\mathcal{E}$ *be a complicial exact category with weak equivalences and duality satisfying* $(*)$*. Then forgetful and hyperbolic functors* $(\mathcal{E},w,\sharp_n) \overset{F}{\to} (H\mathcal{E},w) \overset{H}{\to} (\mathcal{E},w,\sharp_{n+1})$ *defined in* **??** *induce a homotopy fibration up to* π_0

$$GW_{\geq 0}(\mathcal{E},w,\sharp_n) \xrightarrow{F} K(\mathcal{E},w) \xrightarrow{H} GW_{\geq 0}(\mathcal{E},w,\sharp_{n+1}).$$

which is functorial for complicial exact categories with weak equivalences and duality.

Proof. This follows from theorem 2.6, lemma **??**, lemma **??** and the observations that the composition $(\mathcal{E},\sharp_n) \to (\operatorname{Mor}\mathcal{E},\sharp_n) \to H\mathcal{E}$ is the forgetful functor and the composition

$$H\mathcal{E} \quad \xrightarrow{T^{-1}\otimes} \quad H\mathcal{E} \quad \longrightarrow \quad (\operatorname{Mor}\mathcal{E},\sharp_n) \quad \xrightarrow{\text{cone}} \quad (\mathcal{E},\sharp_{n+1})$$

is naturally isometric to the hyperbolic functor. Since $T^{-1}\otimes$ is homotopic to -1 in K-theory, the pair $(F,-H)$, and hence, the pair (F,H), induce homotopy fibrations of spectra. □

2.8. **Lemma** (Karoubi induction). *Let* $G : \mathcal{A} \to \mathcal{B}$ *be a complicial functor between complicial exact categories with weak equivalences and duality satisfying* $(*)$*. Suppose* G *induces an equivalence in* K*-theory and isomorphisms* $GW_0(\mathcal{A},w,\sharp_n) \to GW_0(\mathcal{B},w,\sharp_n)$*,* $n \in \mathbb{Z}/4$*. Then* G *induces an equivalence for* $n \in \mathbb{Z}/4$

$$GW_{\geq 0}(\mathcal{A},w,\sharp_n) \xrightarrow{\simeq} GW_{\geq 0}(\mathcal{B},w,\sharp_n).$$

Proof. By corollary 2.7, G induces a map of exact sequences for $n \in \mathbb{Z}$ and $i \geq 0$

$$\begin{tikzcd}
GW_{i+1}(\mathcal{A}, \sharp_n) \arrow[r] \arrow[d] & K_{i+1}(\mathcal{A}) \arrow[r] \arrow[d] & GW_{i+1}(\mathcal{A}, \sharp_{n+1}) \arrow[r] \arrow[d] & GW_i(\mathcal{A}, \sharp_n) \arrow[r] \arrow[d] & K_i(\mathcal{A}) \arrow[d] \\
GW_{i+1}(\mathcal{B}, \sharp_n) \arrow[r] & K_{i+1}(\mathcal{B}) \arrow[r] & GW_{i+1}(\mathcal{B}, \sharp_{n+1}) \arrow[r] & GW_i(\mathcal{B}, \sharp_n) \arrow[r] & K_i(\mathcal{A}).
\end{tikzcd}$$

By a version of the 5-lemma, an isomorphism $GW_i(G, \sharp_n)$ implies a surjection $GW_{i+1}(G, \sharp_{n+1})$. Moreover, an isomorphism $GW_i(G, \sharp_n)$ and a surjection $GW_{i+1}(G, \sharp_n)$ imply an isomorphism $GW_{i+1}(G, \sharp_{n+1})$. □

2.9. **Theorem** (Invariance under derived equivalences). *Let $G : \mathcal{A} \to \mathcal{B}$ be a complicial functor between complicial exact categories with weak equivalences and duality satisfying $(*)$. Suppose that G induces an equivalence $\mathcal{TA} \to \mathcal{TB}$ of associated triangulated categories. Then G induces an equivalence for $n \in \mathbb{Z}/4$*

$$GW_{\geq 0}(\mathcal{A}, w, \sharp_n) \xrightarrow{\simeq} GW_{\geq 0}(\mathcal{B}, w, \sharp_n).$$

Proof. Corollary **??** implies that for a complicial exact category with weak equivalences and duality $(\mathcal{E}, w)$, the group $K_0^h(\mathcal{E}, w)$ is the co-equalizer of a diagram

$$K_0 \quad w^{-1}(wS_2\mathcal{E})_h \quad \rightrightarrows \quad K_0 \quad w^{-1}(w\mathcal{E})_h,$$

where $K_0(\mathcal{C})$ means "Grothendieck group of the abelian monoid of isomorphism classes of objects in the symmetric monoidal category $\mathcal{C}$". The two maps are induced by the two ways of associating symmetric weak equivalences in $\mathcal{E}$ to a symmetric weak equivalence in $\S_2\mathcal{E}$ as in **??** (**??**).

It is easy to see that a functor $G : \mathcal{A} \to \mathcal{B}$ such that $G : \mathcal{TA} \to \mathcal{TB}$ is an equivalence, also induces an equivalence $\mathcal{T}S_2\mathcal{A} \to \mathcal{T}S_2\mathcal{B}$ of triangulated categories associated with the categories of exact sequences in $\mathcal{A}$ and $\mathcal{B}$. It follows from lemma 2.3, that G induces isomorphisms on both terms in the diagram above, and thus isomorphisms $GW_0(\mathcal{A}, w, \sharp_n) \to GW_0(\mathcal{B}, w, \sharp_n)$, $n \in \mathbb{Z}$. Since, by a theorem of Thomason [TT90, ??], G induces an equivalence in K-theory, we are done by Lemma 2.8. □

2.10. **Theorem.** *Let $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be a sequence of complicial functors between complicial exact categories with weak equivalences and duality satisfying $(*)$. Assume that the associated sequence of triangulated categories $\mathcal{TA} \to \mathcal{TB} \to \mathcal{TC}$ is exact. Then the composition $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ factors through $\mathcal{C}^w$ and the commutative square of (-1)-connected Grothendieck-Witt theory spectra*

$$\begin{tikzcd}
GW_{\geq 0}(\mathcal{A}, w, \sharp_n) \arrow[r] \arrow[d] & GW_{\geq 0}(\mathcal{B}, w, \sharp_n) \arrow[d] \\
0 \simeq GW_{\geq 0}(\mathcal{C}^w, w, \sharp_n) \arrow[r] & GW_{\geq 0}(\mathcal{C}, w, \sharp_n)
\end{tikzcd}$$

is homotopy cartesian up to π_0, and the lower left corner is contractible.

Proof. This follows from the "change of weak equivalence" theorem **??**, and invariance under derived equivalences (theorem 2.9).

□

3. The non-connective Grothendieck-Witt-theory spectrum

Let $(\mathcal{E}, w)$ be a complicial exact category with weak equivalences and duality satisfying $(*)$. We will construct a spectrum $GW(\mathcal{E}, w)$ and a map

$$GW_{\geq 0}(\mathcal{E}, w) \to GW(\mathcal{E}, w)$$

of spectra functorial in $\mathcal{E}$ such that $\pi_i GW_{\geq 0}(\mathcal{E}, w) \to \pi_i GW(\mathcal{E}, w)$ is an isomorphism for $i \geq 0$, and such that the fibration up to π_0 of theorem 2.10 extends to an honest fibration of Grothendieck-Witt theory spectra.

3.1. **Definition.** Define the Witt groups functors $W_0(n)$ as $W_0(n)(\mathcal{A}, w, \sharp, \eta) = W_0(\mathcal{A}, w, \sharp_n, \eta^n)$ and remark that $W_0(n)(\mathrm{Mor}\,\mathcal{A}, w, \sharp, \eta) = 0$...

3.2. **Construction.** We will construct a sequence of functors from complicial exact categories with weak equivalences and duality satisfying $(*)$ to spectra

$$GW_{\geq 0} \to GW_{\geq -1} \to GW_{\geq -2} \to GW_{\geq -3} \to \cdots \tag{14}$$

such that for $n \leq 0$

1. $GW_{\geq n}$ is $n-1$ connected,
2. for $n < 0$, the map $GW_{\geq n+1} \to GW_{\geq n}$ is an isomorphism in degrees $\geq n+1$, and $\pi_n GW_{\geq n} = W_0(n)$,
3. theorem 2.10 holds with "up to π_n" and $GW_{\geq n}$ in place of "up to π_0" and $GW_{\geq 0}$,
4. The map $GW_{\geq n+1}\,\mathrm{Mor} \to GW_{\geq n}\,\mathrm{Mor}$ is an equivalence for $n < 0$,
5. lemma **??** holds for complicial functors F and G, and $GW_{\geq n}$ in place of $GW_{\geq 0}$.

Note that (4) implies that $GW_{\geq n}(\mathcal{A}) \simeq 0$ for $\mathcal{A} = 0$ the category consisting of only a zero object, since in this case $\mathcal{A} \cong \mathrm{Mor}\,\mathcal{A}$, and $GW_{\geq 0}(\mathcal{A}) = 0$.

We have seen that the functor $GW_{\geq 0}$ satisfies (1) - (5); (1) holds by contruction of $GW_{\geq 0}$, (2) is vacuous, (3) is theorem 2.10, (4) is vacuous, and (5) is lemma **??**.

Assume we have constructed the functors $GW_{\geq -l}$ for $l = 0, ..., n \in \mathbb{N}$ satisfying (1) - (5). Let $(\mathcal{A}, w)$ be a complicial exact category with weak equivalences and duality satisfying $(*)$. Consider diagram 2.6 (13), and define $F_{\geq -n}$ and $G_{\geq -n-1}$ as the homotopy fibres (in the category of spectra) of $GW_{\geq -n}$ of the left and right vertical maps in that diagram. More precisely, $F_{\geq -n}(\mathcal{A}, w, \sharp, \eta)$ is the homotopy fibre of

$$GW_{\geq -n}(\mathcal{A}, w, \sharp, \eta) \to GW_{\geq -n}(\mathcal{A}^w, w, \sharp_1, \eta^1),$$

and $G_{\geq -n-1}(\mathcal{A}, w, \sharp, \eta)$ is the homotopy fibre of

$$GW_{\geq -n}(\mathrm{Mor}\,\mathcal{A}, w, \sharp, \eta) \to GW_{\geq -n}(\mathcal{A}, w, \sharp_1, \eta^1).$$

The natural map $F_{\geq -n} \to GW_{\geq -n}$ is an equivalence since $GW_{\geq -n}(\mathcal{A}^w, w, \sharp_1, \eta^1)$ is contractible, by (5) and the remark thereafter. The natural map $F_{\geq n} \to G_{\geq n-1}$, induced by the commutativity of diagram 2.6 (13), is an equivalence up to π_{-n-1}, by (3). Moreover, $G_{\geq -n-1}$ is $-n-2$-connected, by (1), and $\pi_{-n-1} G_{\geq -n-1}$ is the cokernel of

$$\pi_{-n} GW_{\geq -n}(\mathrm{Mor}\,\mathcal{A}, \sharp) \to \pi_{-n} GW_{\geq -n}(\mathcal{A}, \sharp_1)$$

which is $W_0(\mathcal{A}, \sharp_1) = W_0(1)(\mathcal{A}, \sharp)$ for $n = 0$, and $W_0(n)(\mathcal{A}, \sharp_1) = W_0(n+1)(\mathcal{A}, \sharp)$ for $n > 0$ since $W_0(n)(\mathrm{Mor}, \mathcal{A}, \sharp) = 0$. In particular, the map $F_{\geq n}\,\mathrm{Mor} \to G_{\geq n-1}\,\mathrm{Mor}$ is an equivalence. Moreover, theorem 2.10 holds with "up to π_{n-1}" and $G_{\geq n-1}$ in place of "up to π_0" and $GW_{\geq 0}$, by (3) and (4), and the fact that the diagram in theorem 2.10 is homotopy cartesian when $GW_{\geq 0}$ is replaced with $GW_{\geq 0}\,\mathrm{Mor}$ which is (-1)-connected K-theory, by lemma **??**. Finally, (5) holds for $G_{\geq -n-1}$ by functoriality of its construction.

Now we define $GW_{\geq -n-1}$ as the homotopy colimit

$$GW_{\geq -n-1} = \mathrm{hocolim}(GW_{\geq -n} \xleftarrow{\sim} F_{\geq -n} \longrightarrow G_{\geq -n-1}).$$

It comes with a natural map $GW_{\geq -n} \to GW_{\geq -n-1}$. Since the induced map $G_{\geq -n-1} \to GW_{\geq -n-1}$ is an equivalence (as $F_{\geq -n} \to GW_{\geq -n}$ is), the properties (1) - (5) for $GW_{\geq -n-1}$ readily follow from the corresponding properties of $G_{\geq -n-1}$.

3.3. **Definition.** Let $(\mathcal{E}, w, \sharp, \eta)$ be a complicial exact category with weak equivalences and duality satisfying $(*)$. Its *non-connective Grothendieck-Witt theory spectrum* $GW(\mathcal{E}, w, \sharp, \eta)$ is defined to be the colimit

$$GW(\mathcal{E}, w, \sharp, \eta) = \mathrm{colim}_{n\in\mathbb{N}}\, GW_{\geq -n}(\mathcal{E}, w, \sharp, \eta)$$

of the sequence (14). It is functorial for complicial functors of such categories.

The following, proposition 3.4 and theorem 3.5, follow immediately from the construction and discussion in 3.2.

3.4. **Proposition.** *Let $\mathcal{E}$ be a complicial exact category with weak equivalences and duality satisfying $(*)$. Then the natural map*

$$GW_{\geq 0}(\mathcal{E}, w) \to GW(\mathcal{E}, w)$$

is an isomorphism in degrees $i \geq 0$, and in negative degrees we have natural isomorphisms

$$GW_{-n}(\mathcal{E}, w, \sharp) = W_0(\mathcal{E}, w, \sharp_n), \qquad n > 0.$$

□

3.5. **Theorem** (Abstract Localization). *Let $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be a sequence of complicial functors between complicial exact categories with weak equivalences and duality satisfying $(*)$. Assume that the associated sequence of triangulated categories $\mathcal{TA} \to \mathcal{TB} \to \mathcal{TC}$ is exact. Then the composition $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ factors through $\mathcal{C}^w$, the commutative square of Grothendieck-Witt theory spectra*

$$\begin{tikzcd}
GW(\mathcal{A}, w, \sharp) \arrow[r] \arrow[d] & GW(\mathcal{B}, w, \sharp) \arrow[d] \\
0 \simeq GW(\mathcal{C}^w, w, \sharp) \arrow[r] & GW(\mathcal{C}, w, \sharp)
\end{tikzcd}$$

is homotopy cartesian, and the lower left corner is contractible. □

3.6. **Karoubi periodicity** (or the fundamental theorem of hermitian K-theory). Let $\mathcal{E}$ be a complicial exact category with weak equivalences and duality. It is easy to see (and implicit in the proof of theorem 2.6) that the sequence

$$(\mathcal{E}, w, \sharp_n) \xrightarrow{X \mapsto 1_X} (\mathrm{Mor}\,\mathcal{E}, w, \sharp_n) \xrightarrow{\mathrm{cone}} (\mathcal{E}, w, \sharp_{n+1})$$

induces an exact sequence of associated triangulated categories. Therefore, theorem 3.5 implies (in the same way as theorem 2.6 implies corollary 2.7) a homotopy fibration of Grothendieck-Witt theory spectra

$$GW(\mathcal{E}, w, \sharp_n, \eta^n) \xrightarrow{F} K(\mathcal{E}, w) \xrightarrow{H} GW(\mathcal{E}, w, \sharp_{n+1}, \eta^{n+1}) \tag{15}$$

for $(\mathcal{E}, w, \sharp, \eta)$ a complicial exact category with weak equivalences and duality satisfying $(*)$.

For $\varepsilon \in \{\pm 1\}$, write ${}_{\varepsilon}GW(\mathcal{A}, n)$ for $GW(\mathcal{A}, w, \sharp_n, \varepsilon\eta^n)$, and ${}_{\varepsilon}GW(\mathcal{A})$ for ${}_{\varepsilon}GW(\mathcal{A}, 0)$. Following Karoubi, one defines new theories ${}_{\varepsilon}U(\mathcal{A})$ and ${}_{\varepsilon}V(\mathcal{A})$ as the homotopy fibres of hyperbolic and forgetful functors

$$K(\mathcal{A}) \xrightarrow{H} {}_{\varepsilon}GW(\mathcal{A}) \quad \text{and} \quad {}_{\varepsilon}GW(\mathcal{A}) \xrightarrow{F} K(\mathcal{A}).$$

From the homotopy fibration (15), we see that

$${}_{\varepsilon}U(\mathcal{A}) \simeq {}_{\varepsilon}GW(\mathcal{A}, -1) \quad \text{and} \quad {}_{\varepsilon}V(\mathcal{A}) \simeq \Omega\ {}_{\varepsilon}GW(\mathcal{A}, 1).$$

It follows from lemma 1.21 with $L = T$ (and $\mathrm{sgn}_T = -1$) that ${}_{\varepsilon}GW(\mathcal{A}, -1) \cong {}_{-\varepsilon}GW(\mathcal{A}, 1)$ so that we obtain a homotopy equivalence

$${}_{-\varepsilon}V(\mathcal{A}) \simeq \Omega\ {}_{\varepsilon}U(\mathcal{A}).$$

For $\mathcal{A}$ the category of finitely generated projective R-modules (and the stable version of higher Grothendieck-Witt theory as defined in section **??** instead of GW), this is Karoubi's fundamental theorem [Kar80].

4. NEGATIVE GROTHENDIECK-WITT GROUPS ARE TRIANGULAR WITT- AND $\mathbb{L}$-GROUPS

4.1. **The Balmer-Walter definition.** Let $\mathcal{A}$ be a complicial exact category with weak equivalences and duality. Recall from 2.5 (12) the cone functor whose associated form gives, for $n = -1$, a symmetric weak equivalence $\mathrm{Cone}(u) \to \mathrm{Cone}(u^{\sharp_{-1}})^{\sharp}$. If $u = u^{\sharp_{-1}}\eta^{-1}$, we obtain a symmetric weak equivalence $\varphi : \mathrm{Cone}(u) \to (\mathrm{Cone}(u))^{\sharp}$. If $\mathcal{A}$ is $\mathbb{Z}[\frac{1}{2}]$-complicial, it is shown in [Bal00] that the isometry class of $(\mathrm{Cone}(u), \varphi)$ in $(\mathcal{T}\mathcal{A}, \sharp)$ only depends on the image of u in $\mathcal{T}\mathcal{A}$, and that $(\mathrm{Cone}(u), \varphi)$ can be constructed entirely within the framework of a triangulated category with duality.

In [Wal03], the *Grothendieck-Witt group* $GW(\mathcal{T})$ of a $\mathbb{Z}[\frac{1}{2}]$-linear triangulated category with dualtiy $\mathcal{T}$ is defined to be the abelian group freely generated by symmetric spaces $[X, \varphi]$ in $\mathcal{T}$ modulo the relations $[X, \varphi] + [Y, \psi] = [(X, \varphi) \perp (Y, \psi)]$ and $[\mathrm{Cone}(u), \varphi] = [H(X)]$ for any symmetric map $u : X \to X^{\sharp_{-1}}$ in $\mathcal{T}$. If $\mathcal{T}$ is the triangulated category $\mathcal{T}\mathcal{A}$ associated with a $\mathbb{Z}[\frac{1}{2}]$-complicial exact category with duality $\mathcal{A}$, we can assume u to be symmetric in $(\mathcal{A}, \sharp_{-1})$ as any symmetric map in $\mathcal{T}\mathcal{A}$ can be lifted (up to isometry in $\mathcal{T}\mathcal{A}$) to a symmetric map in $\mathcal{A}$.

4.2. **Lemma.** *Let $\mathcal{A}$ be a $\mathbb{Z}[\frac{1}{2}]$-complicial exact category with weak equivalences and duality. Then the map of abelian groups is well-defined and an isomorphism*

$$GW_0(\mathcal{A}, w) \to GW(\mathcal{T}\mathcal{A}) : [X, \varphi] \mapsto [X, \varphi]$$

Proof. Let $(\mathcal{E}, w)$ be a complicial exact category with strong duality. A symmetric space (X, φ) in $\mathcal{E}$ defines a symmetric space in $\mathcal{T}\mathcal{E}$. If $L \subset X$ is a Lagrangian, then $[X, \varphi] = [H(L)]$ in $GW(\mathcal{T}\mathcal{E})$ ([**?**, 2.11]). So we obtain a well-defined map $GW_0(\mathcal{E}) \to GW(\mathcal{T}\mathcal{E}) : [X, \varphi] \mapsto [X, \varphi]$ which is obviously trivial on $GW_0(\mathcal{E}^w)$.

Since by remark **??** the group $GW_0(\mathcal{A}, w)$ is the cokernel of $GW_0(\mathcal{A}^{\mathrm{str},w}) \to GW_0(\mathcal{A}^{\mathrm{str}})$, we obtain a well-defined map $GW_0(\mathcal{A}, w) \to GW(\mathcal{T}\mathcal{A}^{\mathrm{str}}) = GW(\mathcal{T}\mathcal{A}) : [X, \varphi] \mapsto [X, \varphi]$. By lemma 2.3, this map is surjective.

To see that this map is also injective, we construct its inverse. Consider a symmetric map $u : X \to X^{\sharp_{-1}}$ in $(\mathcal{A}, \sharp_{-1})$. The definition of the cone functor in 2.5 yields a symmetric weak equivalence of exact sequences

$$\begin{tikzcd}
{[X, T^{\sharp}]} \arrow[r, tail] \arrow[d, "1"] & \mathrm{Cone}(u) \arrow[r, two heads] \arrow[d, "\varphi"] & TX \arrow[d, "T\eta"] \\
{[X, T^{\sharp}]} \arrow[r, tail] & (\mathrm{Cone}(u))^{\sharp} \arrow[r, two heads] & T(X^{\sharp\sharp})
\end{tikzcd}$$

Using the isomorphism of lemma 2.3 and corollary **??**, we obtain a well-defined surjective map of abelian groups $GW(\mathcal{TA}, \sharp) \to GW_0(\mathcal{A}, w, \sharp)$ such that the composition $GW_0(\mathcal{A}, w) \to GW(\mathcal{TA}) \to GW_0(\mathcal{A}, w)$ is the identity map.

□

4.3. **Remark** (Extension of Walter's exact sequence). Let $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be a sequence of complicial functors between $\mathbb{Z}[\frac{1}{2}]$-complicial exact categories with duality such that the associated sequence of triangulated categories $\mathcal{TA} \to \mathcal{TB} \to \mathcal{TC}$ is exact. In [Wal03], Walter extends Balmer's localization exact sequence of Witt-groups [Bal00] to an exact sequence

$$GW(\mathcal{TA}, \sharp_n) \to GW(\mathcal{TB}, \sharp_n) \to GW(\mathcal{TC}, \sharp_n) \to W(\mathcal{TA}, \sharp_{n+1}) \to W(\mathcal{TB}, \sharp_{n+1}) \to \cdots$$

which continues to the right with triangular Witt groups. By theorem 3.5 and lemma 4.2, this sequence now extends to the left with higher Grothendieck-Witt groups. In fact, it is the negative degree part of the long exact sequence of homotopy groups associated with the homotopy fibre square of theorem 3.5.

4.4. **The functor $F^{(\infty)}$.** Let F be a functor from complicial exact categories with weak equivalences and duality. Besides the Grothendieck-Witt theory functor GW introduced in section 3, we have in mind the functors $K_{h\mathbb{Z}/2}$ and $K^{h\mathbb{Z}/2}$ – the homotopy orbit and homotopy fixed point spectra of K-theory under a $\mathbb{Z}/2$-action explained below. Let $F^{(1)}$ be the homotopy cofibre of $F\,\mathrm{Mor}(-1) \to F$ induced by the cone map 12. This comes with a map $F \to F^{(1)}$. Now set $F^{(n+1)} = (F^{(n)})^{(1)}$ and $F^{(0)} = F$. Then we have a sequence of functors

$$F = F^{(0)} \to F^{(1)} \to F^{(2)} \to F^{(3)} \to \cdots \to F^{(\infty)} = \mathrm{colim}_n\, F^{(n)}$$

whose (homotopy) colimit is defined to be $F^{(\infty)}$. We will calculate $F^{(\infty)}$ for F the functors GW, $K_{h\mathbb{Z}/2}$ and $K^{h\mathbb{Z}/2}$.

4.5. **Reminder on $\mathbb{L}$-theory.**

4.6. **Lemma.** *On the category of complicial exact categories with weak equivalences and duality satisfying* $(*)$*, there is a natural equivalence*

$$GW^{(\infty)} \simeq \mathbb{L}$$

where $\mathbb{L}$ *denotes Ranicki's* $\mathbb{L}$*-theory spectrum.*

5. RELATION TO $\mathbb{Z}/2$-EQUIVARIANT HOMOTOPY THEORY

5.1. **G-equivariant homotopy theory.** Let G be a finite group...

5.1.1. *Homotopy limit and colimit spectra.*

5.1.2. *Tate spectra.*

5.1.3. *Example: Eilenberg McLane spectra.*

5.2. **The involution on the K-theory spectrum.** Let $(\mathcal{E}, w, *, \eta)$ be an exact category with weak equivalences and duality. We will construct an action of the 2-element group $\mathbb{Z}/2$ on the ((-1)-connected) K-theory spectrum $K(\mathcal{E}, w)$ of $(\mathcal{E}, w)$. For related constructions, see [Vog85], [**?**]. The construction is in three steps. First, we construct the action on the K-theory $K(\mathcal{A})$ of an additive category $\mathcal{A}$, then on the K-theory $K(\mathcal{E})$ of an exact category, and finally on the K-theory spectrum $K(\mathcal{E}, w)$ of an exact category with weak equivalences. Recall from lemma **??**, that we can assume the duality $*$ on $\mathcal{E}$ to be strict, that is, $\eta = id$, by replacing $(\mathcal{E}, w)$ with $(\mathcal{E}^{str}, w)$.

The involution for additive categories. Let $(\mathcal{C}, *)$ be a category with strict duality $(** = id)$. The functor defined on objects and morphisms by

$$\sigma : i\mathcal{C} \to i\mathcal{C} : \quad A \mapsto A^*, \quad a \mapsto (a^*)^{-1} \tag{16}$$

satisfies $\sigma \circ \sigma = id$, and thus defines a $\mathbb{Z}/2$-action on the category $i\mathcal{C}$. For an additive category with strict duality $(\mathcal{A}, *)$, the symmetric monoidal category $i\mathcal{A}$ has therefore a $\mathbb{Z}/2$-action defined by (16). In order to obtain a $\mathbb{Z}/2$-action on the associated spectrum, we employ the theory of Γ-spaces [**?**].

Let Γ^{op} be the category of pointed finite sets. We will construct a Γ-object

$$\underline{\mathcal{A}} : \Gamma^{op} \to \mathrm{cat}$$

in the category of small categories as follows. For a finite pointed set S (with base point denoted by 0), the category $\underline{\mathcal{A}}(S)$ has objects pairs (A, α) where $\alpha : \mathbb{N} \to S$ is a map such that $\alpha(i) = 0$ for all but finitely many $i \in \mathbb{N}$, and where $A : \mathbb{N} \to Ob\underline{\mathcal{A}}$ is a map such that $A_i = 0$ whenever $\alpha(i) = 0$. A map $f : (A, \alpha) \to (B, \beta)$ in $\underline{\mathcal{A}}(S)$ is a collection $f = (f_{i,j})$ of maps $f_{i,j} : A_j \to B_i$ of maps in $\underline{\mathcal{A}}$ such that $f_{i,j} = 0$ whenever $\alpha(i) \neq \beta(j)$. Composition is matrix multiplication, that is, $(f \circ g)_{i,j} = \Sigma_k f_{i,k} \circ g_{k,j}$. For a pointed map $\theta : S \to T$, the functor

$$\theta_* : \underline{\mathcal{A}}(S) \to \underline{\mathcal{A}}(T)$$

sends an object (A, σ) and a map f as above to the object $(\theta A, \theta \circ \sigma)$ and to the map θf where $(\theta A)_i = A_i$ and $(\theta f)_{i,j} = f_{i,j}$ whenever $\theta\sigma(i), \theta\sigma(j) \neq 0$ and 0 otherwise.

The strict duality $*$ on $\mathcal{A}$ induces a strict duality $*$ on $\underline{\mathcal{A}}(S)$ by $(A, \sigma)^* = (A^*, \sigma)$ with $(A^*)_i = A_i^*$ and $(f^*)_{i,j} = f_{j,i}^*$. Note that $\theta * = * \theta$, for $\theta : S \to T$ as above, so that $\underline{\mathcal{A}}$ is actually a Γ object in the category of small categories with strict duality.

Taking the subcategory of isomorphisms of $\underline{\mathcal{A}}$, we obtain a Γ object $S \mapsto i\underline{\mathcal{A}}(S)$ in the category of small categories with strict dualities which, by (16), is equipped with a map of Γ objects $\sigma : i\underline{\mathcal{A}} \to i\underline{\mathcal{A}}$ satisfying $\sigma \circ \sigma = id$. Taking nerves, we obtain a Γ-object

$$N_* i\underline{\mathcal{A}} : \Gamma^{op} \to \Delta^{op}\,\mathrm{Sets}$$

in the category of simplicial sets which is equipped with an involutive automorphism σ.

5.3. **Lemma.** *On the category of complicial exact categories with weak equivalences and duality satisfying* $(*)$*, there are natural equivalences*

$$(K^{h\mathbb{Z}/2})^{(\infty)} \simeq \hat{\mathbb{H}}(\mathbb{Z}/2, K), \quad \text{and} \quad (K_{h\mathbb{Z}/2})^{(\infty)} \simeq 0.$$

5.4. **Theorem.** *The two squares of spectra below are homotopy cartesian for any complicial exact category with weak equivalences and duality satisfying* $(*)$

$$\begin{tikzcd}
K_{h\mathbb{Z}/2} \arrow[r] \arrow[d] & GW \arrow[r] \arrow[d] & K^{h\mathbb{Z}/2} \arrow[d] \\
* \arrow[r] & \mathbb{L} \arrow[r] & \hat{\mathbb{H}}(\mathbb{Z}/2, K).
\end{tikzcd}$$

6. Cofinality revisited

Let $\mathcal{A} \to \mathcal{B}$ be a map of complicial exact categories with weak equivalences such that $\mathcal{T}\mathcal{A} \to \mathcal{T}\mathcal{B}$ is cofinal, that is, fully faithful and every object of $\mathcal{T}\mathcal{B}$ is a direct factor of an object of $\mathcal{T}\mathcal{A}$. Let $K_0(\mathcal{B}, \mathcal{A})$ be the monoid of isomorphism classes in $\mathcal{T}\mathcal{B}$ under direct sum operation, modulo the submonoid of isomorphisms classes of objects in $\mathcal{T}\mathcal{A}$. There is a canonical map $K_0(\mathcal{B}, w) \to K_0(\mathcal{B}, \mathcal{A}) : [B] \mapsto [B]$ which induces an isomorphism of the cokernel of $K_0(\mathcal{A}, w) \to K_0(\mathcal{B}, w)$ with $K_0(\mathcal{B}, \mathcal{A})$. By a theorem of Thomason [TT90, 1.10.1], there is a homotopy fibration

$$K(\mathcal{A}) \to K(\mathcal{B}) \to K_0(\mathcal{B}, \mathcal{A}) \tag{17}$$

where $K_0(\mathcal{B}, \mathcal{A})$ denotes the Eilenberg-MacLane spectrum whose only homotopy group is in degree 0, where it is $K_0(\mathcal{B}, \mathcal{A})$. Note that the homotopy fibration (17) also follows from the following lemma (for hyperbolic categories) and invariance under derived equivalences of algebraic K-theory.

6.1. **Lemma.** *Let* $(\mathcal{B}, w, \sharp, \eta)$ *be a complicial exact category with weak equivalences and duality, and let* $G \subset K_0(\mathcal{T}\mathcal{B})$ *be a subgroup invariant under the involution. Let* $\mathcal{A} \subset \mathcal{B}$ *be the full subcategory of objects whose class in* $K_0(\mathcal{T}\mathcal{B})$ *lie in* G*. Then* $(\mathcal{A}, w \cap \mathcal{A}, \sharp, \eta)$ *is a complicial exact category with weak equivalences and duality, and the map*

$$GW_n(\mathcal{A}, w) \to GW_n(\mathcal{B}, w)$$

is an isomorphism for $n \geq 1$ *and a monomorphism for* $n = 0$*.*

Proof. It follows from remark **??** and the observation $(\mathcal{A}^{\mathrm{str}})^w = (\mathcal{B}^{\mathrm{str}})^w$ that cofibre of $GW_{\geq 0}(\mathcal{A}, w) \to GW_{\geq 0}(\mathcal{B}, w)$ is equivalent to the cofibre of $GW_{\geq 0}(\mathcal{A}^{\mathrm{str}}, i) \to GW_{\geq 0}(\mathcal{B}^{\mathrm{str}}, i)$. The cofinality theorem **??** for exact categories with duality applied to the cofinal inclusion $\mathcal{A}^{\mathrm{str}} \subset \mathcal{B}^{\mathrm{str}}$ yields the lemma. □

6.2. **Theorem** (Cofinality). *Let* $\mathcal{A} \to \mathcal{B}$ *be a map of complicial exact categories with weak equivalences and duality satisfying* $(*)$*. If* $\mathcal{T}\mathcal{A} \to \mathcal{T}\mathcal{B}$ *is cofinal, then there are homotopy fibrations of spectra*

$$\begin{array}{lclcl}
GW(\mathcal{A}, w) & \to & GW(\mathcal{B}, w) & \to & \mathbb{H}^{\bullet}(\mathbb{Z}/2, K_0) \\
\mathbb{L}(\mathcal{A}, w) & \to & \mathbb{L}(\mathcal{B}, w) & \to & \hat{\mathbb{H}}(\mathbb{Z}/2, K_0)
\end{array}$$

where K_0 *is the* $\mathbb{Z}/2$*-modules* $K_0(\mathcal{T}\mathcal{B}, \mathcal{T}\mathcal{A})$*.*

Proof. Write $GW^?$ and $\mathbb{L}^?$ for the homotopy cofibres of $GW(\mathcal{A},w) \to GW(\mathcal{B},w)$ and $\mathbb{L}(\mathcal{A},w) \to \mathbb{L}(\mathcal{B},w)$. By cofinality in algebraic K-theory (17) and theorem 5.4, we have a homotopy cartesian square

$$\begin{tikzcd}
GW^? \arrow[r] \arrow[d] & \mathbb{H}^{\bullet}(\mathbb{Z}/2, K_0) \arrow[d] \\
\mathbb{L}^? \arrow[r] & \hat{\mathbb{H}}(\mathbb{Z}/2, K_0).
\end{tikzcd}$$

By lemma 6.1, the upper horizontal map is an isomorphism on homotopy groups in positive degrees, since both groups are zero in this range. It follows that the lower horizontal map is an isomorphism on homotopy groups in degrees ≥ 2. Since the lower horizontal map is periodic, it induces an isomorphism in all degrees, and is therefore an equivalence. It follows that the upper horizontal map is also an equivalence. □

6.3. **Remark.** The $\mathbb{L}$-theory fibration of theorem 6.2 is of course not new. It is called "Rothenberg sequence" and is proved, for instance, in [**?**].

Part 3. **Higher Grothendieck-Witt groups of schemes**

7. Vector bundle Grothendieck-Witt groups

Zariski and Nisnevich descent, projective bundle theorem, rigidity, obstruction to excision in triangular Witt theory

8. Coherent Grothendieck-Witt groups

Dfn, localization, Poincare duality, homotopy invariance, Brown-Gersten spectral sequence, Gersten-conjecture and oriented Chow groups

References

[Bal00] Paul Balmer. Triangular Witt groups. I. The 12-term localization exact sequence. *K-Theory*, 19(4):311–363, 2000.

[Gra76] Daniel Grayson. Higher algebraic K-theory. II (after Daniel Quillen). In *Algebraic K-theory (Proc. Conf., Northwestern Univ., Evanston, Ill., 1976)*, pages 217–240. Lecture Notes in Math., Vol. 551. Springer, Berlin, 1976.

[Kar80] Max Karoubi. Le théorème fondamental de la K-théorie hermitienne. *Ann. of Math. (2)*, 112(2):259–282, 1980.

[Kel94] Bernhard Keller. Deriving DG categories. *Ann. Sci. École Norm. Sup. (4)*, 27(1):63–102, 1994.

[KML71] G. M. Kelly and S. Mac Lane. Coherence in closed categories. *J. Pure Appl. Algebra*, 1(1):97–140, 1971.

[Kob99] Damjan Kobal. K-theory, Hermitian K-theory and the Karoubi tower. *K-Theory*, 17(2):113–140, 1999.

[Ran92] A. A. Ranicki. *Algebraic L-theory and topological manifolds*, volume 102 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge, 1992.

[Sch04] Marco Schlichting. Delooping the K-theory of exact categories. *Topology*, 43(5):1089–1103, 2004.

[Sch08] Marco Schlichting. The Mayer-Vietoris principle for Grothendieck-Witt groups of schemes. *http://www.math.lsu.edu/∼mschlich/research.html*, 2008.

[TT90] R. W. Thomason and Thomas Trobaugh. Higher algebraic K-theory of schemes and of derived categories. In *The Grothendieck Festschrift, Vol. III*, volume 88 of *Progr. Math.*, pages 247–435. Birkhäuser Boston, Boston, MA, 1990.

[Vog85] Wolrad Vogell. The involution in the algebraic K-theory of spaces. In *Algebraic and geometric topology (New Brunswick, N.J., 1983)*, volume 1126 of *Lecture Notes in Math.*, pages 277–317. Springer, Berlin, 1985.

[Wal85] Friedhelm Waldhausen. Algebraic K-theory of spaces. In *Algebraic and geometric topology (New Brunswick, N.J., 1983)*, volume 1126 of *Lecture Notes in Math.*, pages 318–419. Springer, Berlin, 1985.

[Wal03] Charles Walter. Grothendieck-Witt groups of triangulated categories. *preprint*, 2003. www.math.uiuc.edu/K-theory/.

[Wil05] Bruce Williams. *Quadratic K-theory and geometric topology*, chapter in Handbook of algebraic K-theory. Springer, 2005.

Marco Schlichting, Department of Mathematics, Louisiana State University, Baton Rouge, LA

E-mail address: `mschlich@math.lsu.edu`