MATSUMOTO'S THEOREM FOR QUADRATIC FORMS

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Abstract. In this note we prove that for a field F of characteristic $\neq 2$, the map $K_n^{MW}(F) \to K_n^h(F,n)$ from Milnor-Witt K-theory to twisted hermitian K-theory of F is an isomorphism in degrees $n \leq 2$.

Introduction

Let F be a field of characteristic $\neq 2$. Milnor-Witt K-groups $K_n^{MW}(F)$ of F were defined in [Mor04b] in order to give a presentation

$$K_n^{MW}(F) \cong [S^0, {\mathbb{G}_m}^{\wedge n}]_{\mathbb{P}^1}$$

of the group of homotopy classes of A^1-stable maps $S^0 \to {\mathbb{G}_m}^{\wedge n}$ of $\mathbb{P}^1$-spectra [Mor04a]. Hornbostel proved in [Hor05] that Karoubi's hermitian K-theory K^h of smooth affine F-schemes is represented in the A^1-homotopy category by a presheaf of symmetric $\mathbb{P}^1$ spectra. Thus the unit map $S^0 \to K^h$ induces a map $K_n^{MW}(F) \cong [S^0, {\mathbb{G}_m}^{\wedge n}]_{\mathbb{P}^1} \to [S^0, {\mathbb{G}_m}^{\wedge n} \wedge K^h]_{\mathbb{P}^1} \cong [S^n, (\mathbb{P}^1)^{\wedge n} \wedge K^h]_{\mathbb{P}^1}$. It follows from the projective bundle theorem for higher hermitian K-theory that $(\mathbb{P}^1)^{\wedge n} \wedge K^h$ represents the twisted hermitian K-theory functor $X \mapsto K^h(X,n)$. Here $K^h(X,n)$ is the hermitian K-theory spectrum associated with the exact category $\mathrm{Ch}^b(X)$ of bounded chain complexes of locally free sheaves of finite rank on X equipped with quasi-isomorphisms of chain complexes as weak equivalences, and equipped with the duality $\mathrm{Hom}(\ ,O_X[n])$ and its canonical double dual identification. Thus, we obtain a map

$$K_n^{MW}(F) \to K_n^h(F,n) \tag{1}$$

analogous to the map $K_n^M(F) \to K_n(F)$ from Milnor K-groups to Quillen K-groups.

In this note we define the map (1) in terms of generators and relations and prove that it is an isomorphism for $n \leq 2$. For $n \leq 0$, this result follows trivially from the construction of the map. Moreover, the result is known for $n = 1$ (see [Kar73], [BL]), but it seems to be new for $n = 2$.

1. Matsumoto's theorem for quadratic forms

1.1. **Twisted hermitian K-theory groups.** Recall from [Sch], that $K^h(F,n)$ is the hermitian K-theory spectrum associated with the exact category $\mathrm{Ch}^b(F)$ of bounded chain complexes of finite dimensional F-vector spaces equipped with the set of homotopy equivalences of chain complexes as weak equivalences, and equipped with the duality $\sharp_n = \mathrm{Hom}(\ ,F[n])$ and its canonical double dual identification.

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Recall also that tensor product of chain complexes induces associative and unital pairings

$$K^h(F,m) \wedge K^h(F,n) \to K^h(F,m+n)$$

with unit $1 = [F, id] \in K_0^h(F,0) = GW(F)$. In particular, $\bigoplus_{n\in\mathbb{Z}} K_n^h(F,n)$ is an associative ring with unit.

A particular case of the generalized version of Karoubi-periodicity proved in [Sch] is the following.

1.2. **Fundamental Theorem.** *If $\frac{1}{2} \in F$, then the following is a homotopy fibration*

$$K^h(F,n) \overset{forget}{\longrightarrow} K(F) \overset{hyperbolic}{\longrightarrow} K(F,n+1).$$

1.3. *Remark.* The fundamental theorem is a special case of a generalization of Thomason-Throbaugh's localization theorem [TT90], [Sch05] applied to the sequence

$$(\mathrm{Ch}^b(F), \mathrm{quis}, \sharp_n) \overset{X\mapsto id_X}{\longrightarrow} (\mathrm{Mor}\,\mathrm{Ch}^b(F), \mathrm{quis}, \sharp_n) \overset{cone}{\longrightarrow} (\mathrm{Ch}^b(F), \mathrm{quis}, \sharp_{n+1})$$

which induces an exact sequences of associated derived categories with duality, and thus, by [Sch], an associated homotopy fibration of hermitian K-theory spectra. By the hermitian additivity theorem, hermitian K-theory of $(\mathrm{Mor}\,\mathrm{Ch}^b(F), \mathrm{quis}, \sharp_n)$ is K-theory of F. The map $K(F) \to K^h(\mathrm{Mor}\,\mathrm{Ch}^b(F), \mathrm{quis}, \sharp_n)$ sends a chain complex of vector spaces V to the map of chain complexes $0 : V^{\sharp_n} \to V$ equipped with the non-degenerate form given by the canonical double dual identification $1 \to \sharp_n \circ \sharp_n$.

Let $\eta \in K_{-1}^h(F,-1) \cong W(F)$ be the element corresponding to $1 = \delta(1) \in W(F)$ where $\delta : K_n^h(F,n) \to K_{n-1}^h(F,n-1)$ is the boundary map in the long exact sequence associated to the homotopy fibration in the fundamental theorem 1.2.

1.4. **Lemma.** *The element $\eta = \delta(1) \in K_{-1}^h(F,-1)$ is in the center of $\bigoplus_{n\in\mathbb{Z}} K_n^h(F,n)$, that is, $\eta \cup x = x \cup \eta$ for all $x \in K_n^h(F,n)$.*

1.5. *Proof.* By [Sch] the -1-connected cover of $\Omega^{-1}K^h(F,-1)$ is $iS_{2*+1}\,\mathrm{Vect}(F)_h$, where $iS_{2n+1}\,\mathrm{Vect}(F)_h$ denotes the category of hermitian objects in $iS_{2n+1}\,\mathrm{Vect}(F)$. The element $\eta \in \pi_0\Omega^{-1}K^h(F,-1)$ corresponds to the object $(F,1)$ in $iS_1\,\mathrm{Vect}(F)_h = i\,\mathrm{Vect}(F)_h$. For any F-linear additive category with duality $\mathcal{A}$, left cup product $\eta\cup$ with η is induced by the map

$$iS_{2*+1}\,\mathrm{Vect}(F)_h \times i\mathcal{A}_h \overset{\otimes}{\to} iS_{2*+1}\mathcal{A}_h,$$

where $\otimes$ denotes $\otimes_F$. Using this description, we see that $\eta\otimes$ is the inclusion of zero simplices $i\mathcal{A}_h = iS_1\mathcal{A}_h \subset iS_{2*+1}\mathcal{A}_h$. Right cup product $\cup\eta$ with η is induced by the map

$$i\mathcal{A}_h \times iS_{2*+1}\,\mathrm{Vect}(F)_h \overset{\otimes}{\to} iS_{2*+1}\mathcal{A}_h.$$

Thus, $\otimes\eta$ is also the inclusion of zero simplices $i\mathcal{A}_h \subset iS_{2*+1}\mathcal{A}_h$. It follows that for any F-linear additive category with duality $\mathcal{A}$, $\cup\eta = \eta\cup : K_m^h(\mathcal{A},0) \to K_{m-1}^h(A,-1)$. Now, $K_n^h(F,n) = K_m^h(\mathcal{A},0)$ for suitable choices of $\mathcal{A}$ and m. $\square$

We have the following interpretation of the boundary map in Karoubi's fundamental theorem.

1.6. **Lemma.** *The boundary map* $\delta : K^h_n(F,n) \to K^h_{n-1}(F,n-1)$ *from the fundamental theorem 1.2 is cup-product* $x \mapsto \eta \cup x$ *with* η.

1.7. *Proof.* By definition of η we have $\delta(1) = \eta$. So we need to check the formula $\delta(x \cup y) = \delta(x) \cup y$ which follows from the fact that $\otimes(Y, \psi)$ is a map of $\mathbb{Z}$-complicial exact categories with duality for any symmetric space (Y, ψ) (since the action of $\mathrm{Ch}^b(\mathbb{Z})$ is on the left). More precisely, the following diagram commutes

$$\begin{tikzcd}
K^h(F,m) \wedge K^h(F,n) \arrow[r] \arrow[d] & K^h(\mathrm{Mor}\,\mathrm{Ch}^b F, \sharp_m) \wedge K^h(F,n) \arrow[r] \arrow[d] & K^h(F,m+1) \wedge K^h(F,n) \arrow[d] \\
K^h(F,m+n) \arrow[r] & K^h(\mathrm{Mor}\,\mathrm{Ch}^b F, \sharp_{m+n}) \arrow[r] & K^h(F,m+n+1),
\end{tikzcd}$$

where the vertical arrows are cup products.

If follows that $\delta(y) = \delta(1 \cup y) = \delta(1) \cup y = \eta \cup y$. □

1.8. *Remark.* This answers a question of Hornbostel in [Hor05].

1.9. $K^h_1(F,1)$ **by generators and relations.** By the fundamental theorem 1.2, $\Omega K^h(F,1)$ is the homotopy fibre of the forgetful map $K^h(F,0) \to K(F)$. Thus, $\Omega K^h(F,1)$ is Karoubi's V-theory [Kar80], that is, $V_n(F) = K^h_{n+1}(F,1)$.

As π_0 of the homotopy fibre in K-theory associated with the cofinal forgetful functor $\varphi : i\,\mathrm{Vect}(F)_h \to i\,\mathrm{Vect}(F)$ from the category $i\,\mathrm{Vect}(F)_h$ of finite dimensional vector spaces equipped with a non-degenerated symmetric bilinear form and maps the isometries, to the category $i\,\mathrm{Vect}(F)$ of finite dimensional vector spaces with maps the isomorphisms, we have an explicit description of $K(\varphi) = V_0(F) = K^h_1(F,1)$ in terms of generators and relations as follows [Kar73], [Bas68, VII§5], [Kar78, II.2.13], [BL].

Consider symbols (V, q_1, q_2) where V is a finite dimensional F-vector space and q_1, q_2 are non-degenerate quadratic forms on V. We say that (V, q_1, q_2) is isomorphic to (V', q'_1, q'_2) if there is an isomorphism $f : V \to V'$ such that $q'_i = q_i \circ f$, $i = 1, 2$. The group $V_0(F) = K^h_1(F,1)$ is generated by isomorphism classes $[V, q_1, q_2]$ of symbols (V, q_1, q_2) modulo the relation $[V, q_0, q_1] + [V, q_1, q_2] + [V, q_2, q_0] = 0$.

1.10. **Explicit products in low degrees.** Cup products

$$K^h_m(F,m) \otimes K^h_n(F,n) \overset{\cup}{\to} K^h_{m+n}(F,m+n)$$

have the following description in low degrees.

- $m = 1, n = 0$
$V_0(F) \otimes GW(F) \to V_0(F) : [V, q_1, q_2] \otimes [W, r] \mapsto [V \otimes W, q_1 \otimes r, q_2 \otimes r]$

$m = 0, n = 1$
$GW(F) \otimes V_0(F) \to V_0(F) : [W, r] \otimes [V, q_1, q_2] \mapsto [W \otimes V, r \otimes q_1, r \otimes q_2]$

- $m = 1, n = -1$
$V_0(F) \otimes W(F) \to GW(F) : [V, q_1, q_2] \otimes [W, r] \mapsto [V \otimes W, q_1 \otimes r] - [V \otimes, q_2 \otimes r]$ which is well-defined since it is zero when $[W, r]$ is hyperbolic.

- $m = -1, n = 1$
$W(F) \otimes V_0(F) \to GW(F) : [W, r] \otimes [V, q_1, q_2] \mapsto [W \otimes V, r \otimes q_1] - [W \otimes V, r \otimes q_2]$

- $m \leq 2, n = \leq 1$ are the usual products of forms.

1.11. *Proof.* The cases $m = 1, n = 0$ and $m = 0, n = 1$ are obvious. The cases $m = 1, n = -1$ and $m = -1, n = 1$ follow from $m = 1, n = 0$ and $m = 0, n = 1$ and the fact that $\delta = \cup\eta = \eta\cup : GW(F) \to W(F)$ is surjective. □

1.12. **The map $K_n^{MW}(F) \to K_n^h(F,n)$.** Recall [Mor04b] that the ring of Milnor-Witt K-groups $K_*^{MW}(F)$ is the (associative and unital) graded ring $\bigoplus_{n\in\mathbb{Z}} K_n^{MW}(F)$ with generators $[a]$ in degree 1 for every $a \in F^\times$, and a symbol η in degree -1 subject to the relations

1. $[ab] = [a] + [b] + \eta[a][b]$ for $a, b \in F^\times$
2. $[a][1-a] = 0$ for $a \in F^\times - \{1\}$
3. $\eta(\eta \cdot [-1] + 2) = 0$
4. $\eta[a] = [a]\eta$ for $a \in F^\times$.

We define the map $K_n^{MW}(F) \to K_n^h(F,n)$ by sending $[a]$ to $[F, a, 1] \in V_0(F) = K_1(F,1)$ and η to $\eta \in K_{-1}(F,-1) = W(F)$ corresponding to the unit element in $W(F)$. We need to check that the four relations hold. To see relation (1), the element $\eta[a][b]$ is sent to $\eta[F,a,1][F,b,1] = ([F,a] - [F,1])[F,b,1] = [F,ab,a] - [F,b,1] = [F,ab,1] - [F,a,1] - F[b,1]$ which is the image of $[ab] - [a] - [b]$. Relation (4) follows from 1.4. To see relation (3), the element $\eta(\eta \cdot [-1] + 2)$ is sent to $\eta(\eta\cdot[F,-1,1]+2) = \eta([F,1]-[F,-1]+2) = \eta([F,1]+[F,-1]) = 0$ since $[F,1]+[F,-1]$ is trivial in $W(F)$. Relation (2) follows from Hornbostel's representation theorem [Hor05].

1.13. **Lemma.** *The map $K_n^{MW}(F) \to K_n^h(F,n)$ is an isomorphism for $n \leq 0$.*

1.14. *Proof.* By [Mor04b], [Mor04a], $GW(F) \to K_0^{MW}(F) : [F,a] \mapsto 1 + \eta \cdot [a]$ is an isomorphism. It follows that the map $K_0^{MW}(F) \to K_0^h(F,0) \cong GW(F)$ is an isomorphism. For $n < 0$, the map $K_n^{MW}(F) \to K_n^h(F,n)$ is an isomorphism because $\delta = \eta\cup : K_n^h(F,n) \to K_{n-1}^h(F,n-1)$ is an isomorphism (by the fundamental theorem and the fact that $K_n(F) = 0$) and because $\eta : K_n^{MW}(F) \to K_{n-1}^{MW}(F)$ is an isomorphism. □

Since $h\eta = 0$ with $h = \eta \cdot [-1] + 2$, multiplication with h induces a map $H : \operatorname{coker}(\eta) = K_n^M(F) \to K_n^{MW}(F)$ which is $\times 2$ when composed with $K_n^{MW}(F) \to K_n^M(F)$, and which is 0 when composed with $K_n^{MW}(F) \to I^n = K_n^{MW}(F)/h$.

1.15. **Lemma.** *The following sequence is exact*

$$K_{n+1}^M(F) \xrightarrow{H} K_{n+1}^{MW}(F) \xrightarrow{\eta} K_n^{MW}(F) \to K_n^M(F) \to 0. \tag{2}$$

1.16. *Proof.* Consider two ways of calculating the cohomology of the cone of the map of complexes

$$\begin{tikzcd}
0 \arrow[r] & K_{n+1}^{MW}(F) \arrow[r, "\eta"] \arrow[d, two heads] & K_n^{MW}(F) \arrow[r] \arrow[d, two heads] & K_n^M(F) \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & I^{n+1}(F) \arrow[r] & I^n(F) \arrow[r] & K_n^M/2(F) \arrow[r] & 0
\end{tikzcd}$$

On one hand both rows are exact, except at $K_{n+1}^{MW}(F)$, so that the total cohomology is $\ker\eta$. On the other hand, the map of complexes is surjective, and the right hand square is cartesian, so that the total cohomology is the kernel of the left vertical map, which is also $\ker[K^M_{n+1}(F) \to K^M_{n+1}/2(F)]$, by the pull-back definition of $K^{MW}(F)$. ☐

1.17. **Lemma.** *The composition* $K^h(F,n) \xrightarrow{H} K(F) \xrightarrow{F} K^h(F,n)$ *of forgetful and hyperbolic functor is cup product* $h\cup$ *with* $h = \eta \cdot [-1] + 2$.

1.18. *Proof.* As in the proof of 1.6, we see that $HF(x \cup y) = HF(x) \cup y$. Now, $HF(y) = HF(1 \cup y) = HF(1) \cup y = h \cup y$. ☐

1.19. **Lemma.** *The natural maps from Milnor- and Milnor-Witt* K*-theory to* K*-theory and hermitian* K*-theory induces a map of exact sequences*

$$\begin{tikzcd}
K^M_{n+1}(F) \arrow[r, "H"] \arrow[d] \arrow[dr, phantom, "(a)"] & K^{MW}_{n+1}(F) \arrow[r, "\eta"] \arrow[d] \arrow[dr, phantom, "(b)"] & K^{MW}_n(F) \arrow[r] \arrow[d] \arrow[dr, phantom, "(c)"] & K^M_n(F) \arrow[r] \arrow[d] \arrow[dr, phantom, "(d)"] & 0 \arrow[d] \\
K_{n+1}(F) \arrow[r, "H"] & K^h_{n+1}(F, n+1) \arrow[r, "\delta"] & K^h_n(F,n) \arrow[r, "F"] & K_n(F) \arrow[r, "H"] & K^h_n(F, n+1)
\end{tikzcd}$$

1.20. *Proof.* The rows are exact by 1.2 and 1.15. Commutativity of (b) follows from 1.6. We check commutativity of (a). The natural maps from Milnor-Witt to hermitian K-theory induce a map chain complexes from $0 \to K^{MW}_{n+1}(F) \xrightarrow{\eta} K^{MW}_n(F) \xrightarrow{h} K^{MW}_n(F) \to 0$ to $0 \to K^h_{n+1}(F, n+1) \xrightarrow{\eta\cup} K^h_n(F,n) \xrightarrow{h\cup} K^h_n(F,n) \to 0$. Since $K^M(F) = \operatorname{coker}(\eta)$, and $h = HF$ (by lemma 1.17), and since the chain complex $(\eta\cup = \delta, F)$ is exact, commutativity of (a) follows. The same argument shows commutativity of (c) and (d). ☐

1.21. **Theorem.** *The map* $K^{MW}_n(F) \to K^h_n(F,n)$ *is an isomorphism for* $n \leq 2$.

1.22. *Proof.* By Lemma 1.13, the map $K^{MW}_n(F) \to K^h_n(F,n)$ is an isomorphism for $n \leq 0$.

By lemma 1.19 with $n = 0$ and a diagram chase, we see that $K^{MW}_1(F) \to K^h_1(F,1)$ is surjective. Cup product with η defines a map $K^h_1(F,1) \to GW(F)$ with image in $I(F)$ (which, by surjectivity, only needs to be checked for elements of the form $[F,a,1]$). Together with the forgetful map $K^h_1(F,1) \to K_1(F) = K^M_1(F)$ this yields a map $K^h_1(F,1) \to K^M_1(F) \times_{k^M_1(F)} I$ such that the composite $K^{MW}_1(F) \to K^h_1(F,1) \to K^M_1(F) \times_{k^M_1(F)} I$ is the isomorphism of [Mor04b]. It follows that $K^{MW}_1(F) \to K^h_1(F,1)$ is also injective, thus an isomorphism.

Repeating the argument of the previous paragraph for $n = 2$ instead of $n = 1$, using the map $\eta^2\cup : K^h_2(F,2) \to GW(F)$ with image in $I^2(F)$, yields the claim for $n = 2$. Of course, this inductive argument breaks down for $n = 3$ because $K^M_3(F) \to K_3(F)$ is not an isomorphism, in general. ☐

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