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★**Vector-valued Laplace transforms and Cauchy problems.**

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FEATURED REVIEW.

The Laplace transform is a widely used and powerful tool in the study of autonomous linear evolution equations

$$u'(t) = Au(t), \quad t \geq 0, \quad u(0) = x, \tag{1}$$

on a Banach space X. To explain the basic idea, we take the Laplace transform of (1) and obtain

$$\widehat{u}(\lambda) = \int_0^\infty e^{-\lambda t} u(t)\,dt \in D(A) \quad \text{and} \quad (\lambda - A)\widehat{u}(\lambda) = x \tag{2}$$

for $\operatorname{Re}\lambda > w$, assuming that A with domain $D(A)$ is closed and u is Laplace transformable. Conversely, if (2) holds and $u \in C(\mathbb{R}_+, X)$, then one can check that u is a mild solution of (1), i.e., the antiderivative of u takes values in $D(A)$ and $A\int_0^t u(s)ds = u(t) - x$ for $t \geq 0$. Note that a mild solution is a classical one if $u \in C^1(\mathbb{R}, X)$.

In order to solve (2), we further suppose that $\lambda - A$ is invertible on a right half-plane (with inverse $R(\lambda, A)$). Then the mild solution $u\colon \mathbb{R}_+ \to X$ of (1) is given by the inverse Laplace transform of the function $r(\lambda) = R(\lambda, A)x$, provided that r belongs to the range of the Laplace transformation.

The first two chapters of this monograph establish the theory of the vector-valued Laplace transform needed for the above approach. Several of the crucial results of Laplace transform theory for scalar-valued functions cannot directly be extended to Banach space valued functions. In these cases, one is usually forced to take into account geometrical properties of the underlying Banach space X—which makes matters more difficult and quite interesting. In the third chapter these results are then applied to the Cauchy problem (1) and the related problem of second order, concentrating on equations which can be solved via (integrated) operator semigroups and cosine families, respectively.

The second part of the book treats the long-term behaviour of solutions, in particular asymptotic stability and almost periodicity. Again (2) is the starting point. In principle, it is possible to determine the behaviour of the resolvent $R(\lambda, A)$ since A is the given object in applications. Thus one may try to derive qualitative properties of u from properties of its transform $\widehat{u} = r$. In other words, one has to prove Tauberian theorems for vector-valued functions. Such results are established in Chapter 4 and applied to evolution equations in the fifth chapter (mostly in the context of C_0-semigroups). In the final three shorter chapters the authors investigate several concrete partial differential equations in the framework of the monograph.

In the past two decades the authors have played an important role in developing the theory presented in the book. Experts know that beyond well-posedness in a classical sense there is a jungle of concepts and theories for the solvability of (1). There are also several competing approaches to the study of asymptotic properties. In my opinion, the authors have chosen a setting which allows for a systematic and accessible treatment and whose scope is still broad enough in view of many applications. In their extensive notes they give full credit to the classics of Laplace transform theory (starting with Euler and Laplace) and to alternative and/or more general approaches to evolution equations (including Lions' distribution semigroups and Da Prato's regularized semigroups). Here they also identify the original sources of the results and discuss additional material and very recent developments. Thus I have decided to give no further references to the literature in this review.

Let us describe the content of the monograph in more detail. The authors start with a brief but comprehensive introduction to the Bochner integral (including the Radon-Nikodým property and convolutions). Also, Riemann-Stieltjes integrals and functions of bounded semivariation are studied in a Banach space setting. The basic properties of the Laplace transformation $\mathcal{L}\colon u \mapsto \widehat{u}$ are treated in detail, without assuming exponential boundedness of u. Special attention is paid to the abscissa of convergence of the Laplace integral and to its (bounded) holomorphic extendability, as well as to operational properties of $\mathcal{L}$. The uniqueness of the Laplace transform, an approximation result for $\mathcal{L}$, and the Post-Widder inversion formula are established in Section 1.7. Further, basic facts on Fourier transforms are recalled and the Plancherel and Paley-Wiener theorems for Hilbert space valued functions are proved. The first chapter is completed by a treatment of elementary properties of the Laplace-Stieltjes transform $\widehat{dF}(\lambda) = \int_0^\infty e^{-\lambda t} dF(t)$ for functions F of locally bounded semivariation.

The second chapter is devoted to deeper theorems describing the range of the Laplace(-Stieltjes) transformation and to inversion formulas. In particular, it is shown that the Laplace-Stieltjes transformation $\mathcal{L}_S$ is an isometric isomorphism between the spaces of Lipschitz functions $F\colon \mathbb{R}_+ \to X$

vanishing at 0 and

$$C_W((0,\infty),X) = \left\{ r \in C^\infty((0,\infty),X)\colon \sup_{\lambda>0} \frac{\lambda^{k-1}}{k!}\, \|r^{(k)}(\lambda)\| < \infty,\ \forall\, k \in \mathbb{N}_0 \right\},$$

both spaces endowed with their natural norm. Moreover, the Laplace transformation $\mathfrak{L}\colon L^\infty(\mathbb{R}_+,X) \to C_W((0,\infty),X)$ is an isometric isomorphism if and only if X has the Radon-Nikodým property (i.e., every X-valued Lipschitz function is differentiable a.e., which holds e.g. if X is reflexive). Here and on several other occasions the vector-valued setting leads to substantial new difficulties compared to the classical results due to Doetsch, Widder, and others. Combined with the operational properties of the Laplace(-Stieltjes) transformation, the isomorphism theorems then imply inversion formulas for $\mathfrak{L}$ and $\mathfrak{L}_S$. In principle, one can now decide whether a function r is contained in the range of $\mathfrak{L}$ or $\mathfrak{L}_S$. But of course it is hard to check that r belongs to $C_W((0,\infty),X)$. Fortunately, there are some cases where one has sufficient conditions which are easier to verify and which arise naturally in applications, namely, decay estimates of r (or r') on a sector larger than (or equal to) a half-plane or complete monotonicity of r (i.e., $(-1)^n r^{(n)} \geq 0$ on (w,∞) for all $n \in \mathbb{N}_0$). (If positivity is involved, it is always assumed that X is an ordered Banach space with a normal cone.) The first approach involves contour integral arguments, whereas the second one amounts to Banach space versions of Bernstein's theorem (which again depend on geometric properties of X).

The third chapter is the core of the monograph (and also the longest one). First, the authors quickly establish the basic properties of C_0-semigroups T, emphasizing Laplace transforms. For instance, the generator of T is defined as the operator A such that $R(\lambda,A) = \widehat{T}(\lambda)$. A k-times integrated semigroup S is a strongly continuous, Laplace transformable function $S\colon \mathbb{R}_+ \to \mathcal{L}(X)$ such that there is a generator A with $R(\lambda,A) = \lambda^k \widehat{S}(\lambda)$ for $\lambda > w$ for some $k \in \mathbb{N}_0$ and $w \in \mathbb{R}$. The basic theory of these objects is developed in the spirit of C_0-semigroups. For instance, it is shown that $u = S(\cdot)x$ is a classical solution of (1) if $x \in D(A^{k+1})$. Inhomogeneous evolution equations are also treated. From the results of the second chapter there immediately follow Hille-Yosida-type characterizations of generators of (integrated) semigroups. The authors then present the standard results on dissipative and holomorphic semigroups, as well as some approximation and perturbation theorems. Some of these facts are direct consequences of the Laplace transform theory. Separate sections are devoted to so-called Hille-Yosida operators (which satisfy the resolvent estimates to generate a C_0-semigroup but may have a non-dense domain) and to operators such that $R(\lambda,A)$ is positive for $\lambda > w$. It is proved that such operators generate locally Lipschitz continuous once and twice integrated semigroups, respectively. As another source of examples the authors study the operator iA where A generates a holomorphic semigroup of angle $\pi/2$. For instance, $i\Delta$ generates a k-times integrated semigroup on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, if and only if $k \geq n\,|\frac{1}{2} - \frac{1}{p}|$. Finally, cosine families and the second-order problem $u''(t) = Au(t)$ are treated. Besides standard results, it is shown that the usual reduction matrix generates a once integrated semigroup on $X \times X$ if and only if A generates a cosine family if and only if the matrix generates a C_0-semigroup on a phase space $V \times X$.

At the beginning of Chapter 4 the authors investigate the relationship between several convergence properties of a function $f\colon \mathbb{R}_+ \to X$ and its antiderivative. It is straightforward to check that if $f(t) \to f_\infty$ as $t \to \infty$, then $(1/t)\int_0^t f(s)ds \to f_\infty$ as $t \to \infty$ (Cesàro convergence), which in turn implies $\lambda\widehat{f}(\lambda) \to f_\infty$ as $\lambda \to 0$ (Abel convergence). The converse (so-called Tauberian) implications are wrong in general, but hold under additional conditions like boundedness, slow oscillation, positivity, or monotonicity of f. Such results are then used to derive classical theorems for power and Fourier series. Moreover, the Tauberian theorems are employed to investigate mean ergodicity of strongly continuous and Lipschitz continuous once integrated semigroups. In the remainder of the chapter spectral and ergodicity conditions play a dominant role. Since the various notions are somewhat technical (but presented very clearly in the book), my account becomes sketchier. The first major theorem, proved by an ingenious contour argument, says that the antiderivative of f converges to $\widehat{f}(0)$ provided that f is bounded, that $\widehat{f}$ can be extended holomorphically to $i\mathbb{R}$ with the exception of at most one point $i\eta \neq 0$, and that $\int_0^t e^{i\eta s} f(s)\,ds$ is bounded for $t \geq 0$. This result has several important consequences, among them continuous-time versions of the Gel$'$fand and Katznelson-Tzafriri theorems. Then the focus shifts to (asymptotic) almost periodic functions on $\mathbb{R}$ (or $\mathbb{R}_+$). The authors give a detailed introduction to these notions. One of the main results states that a function $u \in BUC(\mathbb{R}, X)$ having countable Carleman spectrum is almost periodic if either X does not contain c_0, or u has relatively compact range, or u satisfies a certain ergodicity condition. The proof relies on quite involved functional-analytic methods. The geometric condition on X enters via Kadets' theorem, which is also proved. Finally, using the Fourier transform, it is shown that $f \in BUC(\mathbb{R}_+, X)$ is asymptotically almost periodic (i.e., $f = f_0 + f_1$ where f_1 is almost periodic and $f_0(t) \to 0$ as $t \to 0$) if $\widehat{f}$ can be extended in a distributional sense to $i\mathbb{R}$ with the exception of countably many points.

The next chapter deals with the asymptotic behaviour of solutions to evolution equations. First, many of the standard results on uniform exponential stability of a C_0-semigroup T with generator A are discussed, often based on transform methods. Let us point out two not so standard facts: If $\widehat{Tx}(\lambda)$ has a bounded holomorphic extension for $\operatorname{Re}\lambda > 0$ and some x, then $T(t)R(\mu, A)x$ grows at most linearly as $t \to \infty$. Moreover, a positive semigroup on $L^p(\mu)$, $p \geq 1$, or on $C_0(\Omega)$ is uniformly exponentially stable if the spectrum $\sigma(A)$ of its generator is contained in a half-plane given by $\operatorname{Re}\lambda \leq s < 0$. The authors give a rather elementary approach to a Glicksberg-de Leeuw-type splitting theorem showing that the orbit $T(\cdot)x$ is asymptotically almost periodic if T is bounded and the orbit is relatively compact. (There is also a weak version.) The results of Chapter 4 further imply that if T is bounded and $\sigma(A) \cap i\mathbb{R}$ is countable, then all orbits are asymptotically almost periodic provided that T satisfies an ergodicity condition. If $\sigma(A) \cap i\mathbb{R}$ is countable and the adjoint A^* has no eigenvalues on $i\mathbb{R}$, then the orbits even converge to 0. There are several more general versions of this theorem involving single orbits. Moreover, let A be a closed operator such that $\sigma(A) \cap i\mathbb{R}$ is countable and let $u \in BUC(\mathbb{R}_+, X)$ be a mild solution of the corresponding inhomogeneous problem satisfying a certain ergodicity condition. Then u is asymptotically almost periodic if the same holds for the inhomogeneity f. Finally, based on

Fourier transform methods, it is proved that $u = T * f$ inherits properties of T and f if instead of countability one has non-resonance (i.e., the spectra of A and f on $i\mathbb{R}$ are disjoint).

The results of the first five chapters are illustrated by plenty of (counter-)examples which help a lot to understand the theory. Moreover, in Chapter 3 the (Dirichlet-)Laplacian and its square root are discussed at several points. The final three chapters further treat the heat equation for the Laplacian with inhomogeneous Dirichlet boundary conditions on $\Omega \subset \mathbb{R}^n$, the wave equation, and hyperbolic systems on $\mathbb{R}^n$ with constant coefficients, respectively. To study the heat equation, the Laplacian and the trace operator are put into a matrix operator on $C(\overline{\Omega}) \times C(\partial\Omega)$ which turns out to be resolvent positive. This is the starting point to solve the inhomogeneous problem employing the methods of the previous chapters. One also sees that the mild solution inherits asymptotic properties of the inhomogeneity. The wave equation is solved by form methods and the theory of cosine families. The last chapter, Chapter 8, provides a brief introduction to Fourier multipliers. This theory is then used to show that certain classes of pseudo-differential operators and of systems of partial differential operators generate integrated semigroups on $L^p(\mathbb{R}^n)^N$. This can be applied to Maxwell's or Dirac's equation. Some of the results in Chapter 8 are proved without relying on additional material, whereas others need deeper facts like Mikhlin's theorem.

The above description should indicate the wealth of the results collected in the monograph. As far as I can see, the authors have treated the material in an elegant and efficient way. In particular, the main line of argument is clearly explained throughout, and, despite the wide scope of the book, one always feels a unifying spirit behind the text. The authors have done a good job in keeping the presentation quite self-contained. In addition, many auxiliary results are collected (without proofs) in five appendices. Nevertheless the reader needs a solid background in functional analysis and operator theory to appreciate the book. Some experience with evolution equations and the relevant areas of harmonic analysis will facilitate the understanding of the material.

Several parts of the book can be read separately as excellent introductions to their respective subjects, for instance, the sections on the Bochner integral and on almost periodic functions. Moreover, the monograph gives a condensed and partly unorthodox treatment of the basic theory of C_0-semigroups and some of its advanced subjects. But most importantly, the book should become a standard reference for the Laplace transform and Tauberian theorems in the Banach space setting. Further, it gives a concise account of the existence theory of generalized solutions to Cauchy problems on the prototypical level of integrated semigroups. And it is an impressive treatise on ergodicity, asymptotic stability, and almost periodicity of mild solutions to evolution equations.

Reviewed by *Roland Schnaubelt*