Theories, Analysis and Bounds of the Finite-Support Approximation for the Inverses of Mixing-Phase FIR Systems

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***Abstract*— The inverse system approximation using the finite impulse responses (FIR) is essential to a broad area of signal processing applications. The conventional Wiener filtering techniques based on the least-square approach cannot provide an analytical framework simultaneously governing two crucial problems, namely the selection of model-order and the evaluation of asymptotical error bounds. In fact, the square approximation error induced from the FIR realization of a linear time-invariant system is quite complicated especially for those system transfer functions possessing repeated zeros with large multiplicities. Therefore, in this paper, we establish an isomorphism to characterize the Z-transform pairs. In this mathematical paradigm, we will elaborate the problem of approximating an inverse system or filter with an infinite number of coefficients by an FIR filter and derive the new L_1- and L_2-error bounds between the actual inverse filter and the corresponding approximated FIR. Our new theories, analysis and bounds can be utilized to quantify the appropriate model order for the inverse system approximation which is often needed for signal processing, control, communications, etc.**

***Index Terms*— Inverse Systems, Isomorphism, Approximation Error.** [1]

I. Introduction

The inverse system modeling problem is crucial in a wide variety of signal processing applications, such as seismic data processing, plant control, acoustic echo cancelation, equalization in communications receivers, etc., [1]–[9]. Such an inverse system, often referred as the *inverse filter*, has been drawing a lot of research interest for decades [10]–[14]. The inverse filter, or *equalizer* on some circumstances, can be designed through the least-square approach (*Wiener Filtering*) incorporated with adjustable delays, spectral factorization or regulators to achieve the minimum mean-squared error (MMSE) [10], [14]–[20]. The Hankel-norm approximations and the corresponding optimization algorithms have been proposed for the model-order reduction or the infinite impulse response (IIR) approximations by finite impulse response (FIR) systems [21], [22]. However, the existing literature including [21], [22] has not presented any error (bound) function with respect to the chosen delay for a stabilized inverse approximation of an arbitrary mixing-phase system. Since the chosen delay for the stabilized inverse system approximation determines the model order of the inverse filter or the equalizer, the approximation error quantification versus the model order is of great significance to the practical implementations especially in the real-time signal processing platforms. In this paper, we would like to exploit the theories and mathematical framework for deriving the approximation error bounds versus the chosen delay (model order) associated with the inverse filter corresponding to any mixing-phase system.

According to [10], [23], a linear time-invariant mixing-phase system transfer function $H(z)$ can be factorized as

$$H(z) = H_{max}(z)H_{min}(z), \tag{1}$$

where $H_{max}(z)$, $H_{min}(z)$ denote the maximum-phase and minimum-phase components, respectively. In this paper, we assume that the system does not eliminate or exclusively select any frequency component, i.e., $H(z)$ has neither any zero nor any pole right on the unit circle ($|z| = 1$). Thus, the stabilized inverse system approximation of $H(z)$ can be formulated as

$$G(z) = G_{max}(z)\frac{1}{H_{min}(z)}, \tag{2}$$

where

$$G_{max}(z) \approx \frac{z^{-\triangle}}{H_{max}(z)}, \tag{3}$$

and $G_{max}(z)$ is the FIR approximation of the maximum-phase component inverse accordingly. According to (1)-(3), we obtain

$$G(z)H(z) \approx z^{-\triangle}. \tag{4}$$

For the real-time implementational concern, we restrict $G(z)$ to possess all poles inside the unit circle such that $G(z)$ corresponds to a stable causal filter. There remain two problems in (4), namely, how to quantify the difference (approximation error) between the product $G(z)H(z)$ and the target Z-transform $z^{-\triangle}$, and how to determine the appropriate

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delay $\triangle$ (*stabilizing delay*) given the maximum allowable approximation error once it is quantified. The mean-square error analysis can be found in [24], [25], which have addressed the effects of the single-delay selection for inverse filter on the system performance. However, they do not explicitly quantify the discrepancy between the approximated and the exact inverse systems in terms of the complete impulse response. In this paper, other than the statistical signal analysis, we focus on the inverse system approximation given the known filter. It is an important fundamental problem related to the crucial model-order selection. We will present a new mathematical framework, which leads to the upper bounds of the finite-support approximation for the inverses of mixing-phase FIR systems. This new framework will pave a theoretical foundation for the model-order selection of the inverse filters subject to the controllable approximation errors.

The rest of this paper is organized as follows. In Section II, we provide a thorough discussion on the approximation error evaluation for the stabilized mixing-phase inverse filters. In Section III, we adopt the language from abstract algebra to study the mathematical insights of Z-transform, which can help us address the inverse filtering problem later on. In Section IV, we adopt the isomorphism established in Section III to analyze the problem of approximating an inverse IIR filter as an FIR filter; through the isomorphism, we derive the explicit error bound between the approximated inverse filter and the actual inverse. Our derived error bound function and stabilized inverse system approximation scheme can lead to many potential applications, such as telecommunication equalizer design. In Section V, the derived error bound function is applied for the equalizer design and the equalizer performance with respect to model order is also demonstrated therein. Finally, concluding remarks will be drawn in Section VI.

Notations: The sets of all integers and positive integers are denoted as $\mathbb{Z}$ and $\mathbb{N}$, respectively. The symbol $\equiv$ is used to represent a mathematical definition. A *sequence* is expressed as $\langle a\rangle$ and the corresponding index-n element is represented as $\langle a\rangle_n$ where $n \in \mathbb{Z}$ thereupon. The scalar multiplication of a sequence $\langle a\rangle$ and a scalar ψ is defined as $\langle b\rangle = \psi\langle a\rangle$, where $\langle b\rangle_n \equiv \psi\langle a\rangle_n, \forall n$. A real-valued sequence can be denoted as $\langle a\rangle \geq 0$ if and only if $\langle a\rangle_n \geq 0$, $\forall n$. A sequence can also be expressed as $\langle a\rangle_{n=ind_l(a)}^{ind_u(a)}$ where $ind_l(a)$ and $ind_u(a)$ specify the *starting* and *terminal* indices, respectively. $\mathcal{S}_{\mathcal{I}} \equiv \left\{\langle a\rangle_{n=-\infty}^{+\infty}\right\}$ denotes the set of infinite sequences. When both $ind_l(a)$ and $ind_u(a)$ are finite, $\mathcal{S}_{\mathcal{F}} \equiv \{\langle a\rangle_{n=ind_l(a)}^{ind_u(a)}\}$ denotes the set of *finite-support sequences*, and $\mathcal{S}_{\mathcal{C}} \equiv \{\langle a\rangle_{n=ind_l(a)}^{+\infty}\}$ denotes the set of *causal sequences* while $\mathcal{S}_{\mathcal{A}} \equiv \{\langle a\rangle_{n=-\infty}^{ind_u(a)}\}$ denotes the set of *anti-causal sequences*. $\overline{\lim}$ denotes the *limit superior* operator [26]. $Z(a)$ represents the Z-transform of the sequence $\langle a\rangle$; correspondingly, $Z^{-1}(F(z))$ represents the inverse Z-transform of the rational z-function $F(z)$ if such a sequence exists.

II. Stabilized Approximation of Inverse Mixing-Phase Systems

We consider a linear-time invariant system with the transfer function $H(z)$ specified by a proper rational [2] Z-transform as [23]:

$$H(z) = \frac{P(z)}{Q(z)} = \frac{P(z)}{\prod_{i=1}^{K}(z-p_i)^{m_i}}, \tag{5}$$

where $p_1, p_2, \ldots, p_K$ are the K distinct poles with multiplicities $m_1, m_2, \ldots, m_K$, respectively. According to [23], Eq. (5) can be further expanded as

$$H(z) = \sum_{i=1}^{K}\sum_{l=1}^{m_i}\frac{\beta_{i\,l}}{(z-p_i)^l}. \tag{6}$$

According to Eq. (6), the impulse response $h[n] = Z^{-1}(H(z)), -\infty < n < \infty$, where Z^{-1} is denoted as the inverse Z-transform operator, can be solved as

$$\begin{aligned} h[n] &= \sum_{i\in\mathcal{P}_<}\sum_{l=1}^{m_i}\beta_{i\,l}\, n^l p_i^n \mu[n] \\ &\quad + \sum_{i\in\mathcal{P}_>}\sum_{l=1}^{m_i}\beta_{i\,l}\, n^l p_i^n \mu[-1-n], \end{aligned} \tag{7}$$

where $\mu[n]$ is the unit step sequence as defined in [23] and the two sets $\mathcal{P}_>$ and $\mathcal{P}_<$ are defined as

$$\mathcal{P}_> \equiv \{i : |p_i| > 1, i = 1, 2, \ldots, K\}, \tag{8}$$

and

$$\mathcal{P}_< \equiv \{i : |p_i| < 1, i = 1, 2, \ldots, K\}, \tag{9}$$

respectively. In practice, the finite-support approximation (FIR realization) of the inverse system is often needed to guarantee the bounded-input bounded-output stability. Consequently, the infinite impulse response in (7) has to be truncated thereby. The truncation of the double-sided infinite sequence given by Eq. (7) involves the starting index $ind_l(h_F)$ and the terminal index $ind_u(h_F)$ such that the finite-support approximation $h_F[n]$ of $h[n]$ can be expressed as

$$h_F[n] = \begin{cases} h[n], & ind_l(h_F) \leq n \leq ind_u(h_F) \\ 0, & \text{otherwise} \end{cases}. \tag{10}$$

The exact L_p-norm approximation error $\xi_p(h, h_F)$ can be given by

$$\begin{aligned} \xi_p(h, h_F) &\equiv \sum_{-\infty}^{+\infty}|h[n] - h_F[n]|^p \\ &= \sum_{-\infty}^{ind_l(h_F)-1}|h[n]|^p \\ &\quad + \sum_{ind_u(h_F)+1}^{+\infty}|h[n]|^p. \end{aligned} \tag{11}$$

The exact L_p error defined by Eq. (11) is very complicated to compute especially when there exist some large

[2] However, the techniques presented in this paper can be applied to any rational transfer function.

multiplicities m_i's in Eq. (5) since the approximation error calculation involves the partial fractional decomposition, the inverse Z-transform and the recursion formulae of *arithmetic geometric series* in [27]. Hence, we will simplify this problem by considering the error bounds of $\xi_p(h, h_F)$ for the cases $p = 1$ and $p = 2$ since these two values of p are more relevant to practical applications.

III. Rigorous Theories for Relations between Causal Sequences and z-Functions

In the linear system analysis, the Z-transform converts a discrete-time signal or system, which is a sequence of real or complex numbers, into a complex-valued frequency-domain representation. In this paper, we further elaborate the *morphism* of the Z-transform (transform a sequence in the Hilbert space into a ring of polynomials or rational functions) which stems from the "generating function method" in the probability theory. There are many advantages for us to apply this abstract algebraic approach for discrete-time systems and signals. According to [28], [29], we list some of them here: (a) the morphism studies can help to establish an exact representation for a discrete-time signal or system; (b) the morphism studies can help to solve the difference equations characterizing the linear time-invariant systems and discrete-time signals (sometimes, one may even find a new recurrence relation different from the original difference equation, which provides new insights into the system nature); (c) the morphism studies can help to address the statistical properties of a signal or system (generating functions can lead to extremely quick derivations of various probabilistic aspects associated with the unknown discrete-time signals); (d) the morphism studies can help to manifest the asymptotic behavior or trend of a signal (typically, when one is dealing with a very irregular signal, instead of its exact representation in whatever form, which might be out of the question, we can look for an approximate formula based on the generating function); (e) the morphism studies can help to characterize the variations of a signal, or to infer the rises and falls of its waveform; (f) the morphism studies can help to prove the identities for the essential mathematical operations applied for systems and signals. For example, we have the following identity that

$$\sum_{i=0}^{n} \binom{n}{i} = \binom{2n}{n}, \tag{12}$$

where $n = 0, 1, 2, \cdots$. It becomes much simpler and more illustrative to prove the above identity through the check of the generating functions associated with the sequences at both sides.

In order to derive the L_1 approximation error bound as stated in the previous section, we benefit from the aforementioned merits (a), (c) and (f) of the Z-transform (generating function approach) and the isomorphism between a sequence representing a signal or a system and its corresponding Z-transform in our later proofs to the new theorems. Although the Z-transform is our main tool to derive the L_1 approximation error bound, there exists no literature to establish the corresponding isomorphism. Hence, we dedicate this section to rigorously employing the abstract algebra and to establishing the isomorphism (mathematical equivalence) between causal sequences and the corresponding rational z-functions. The mathematical equivalence between two sets means that some truths (mathematical properties) related to the elements in one set hold for the corresponding elements in the other set. From the language of abstract algebra, the equivalence between two sets can be justified by proving the "isomorphism" between these two sets.

Our discussions will be focused on the stable causal inverse systems (filters) whose impulse responses are causal sequences, i.e., $\langle a \rangle = (\cdots, 0, 0, \alpha_k, \alpha_{k+1}, \alpha_{k+2}, \cdots)$, where $k = ind_l(a) \in \mathbb{Z}$. Without loss of generality, a causal sequence can be denoted as $\langle a \rangle \in \mathcal{S_C}$ or $\langle a \rangle_{n=ind_l(a)}^{+\infty}$ (the filter impulse response sequence $\langle h \rangle$ can also belong to this class of sequences when it is causal). From now on, we treat any impulse response $h[n]$ as a sequence, such that it can also be denoted as $\langle h \rangle$ for our future algebraic manipulation ($\langle h \rangle_n = h[n]$). The causal sequences represent the practical signals and filters. Thus, the (classical) Z-transform of the causal sequence $\langle a \rangle$ is defined as

$$\begin{aligned} Z(a) \;\equiv\;& \alpha_k z^{-k} + \alpha_{k+1} z^{-(k+1)} \\ &+ \alpha_{k+2} z^{-(k+2)} + \cdots \end{aligned} \tag{13}$$

and the power series $Z(a)$ converges under the assumption that

$$\overline{\lim_{n\to\infty}} |\langle a \rangle_n|^{\frac{1}{n}} = \overline{\lim_{n\to\infty}} |\alpha_n|^{\frac{1}{n}} < \infty. \tag{14}$$

Since the sequence $\langle a \rangle$ is causal, there exists an integer $k = ind_l(a)$ such that $\alpha_j = 0$ for $j < k$ and $\alpha_k \neq 0$. Moreover, if $\alpha_{k'} \neq 0$ and $\alpha_j = 0$ for all $j > k' = ind_u(a)$, then such a causal sequence is finite-support. The following lemma characterizes the relationships among $\{ind_l(a*b), ind_u(a*b)\}$ and $\{ind_l(a), ind_u(a), ind_l(b), ind_u(b)\}$ where $*$ denotes the convolutional operation such that

$$\langle c \rangle_n = \langle a \rangle_n * \langle b \rangle_n \equiv \sum_i \langle a \rangle_{n-i} \langle b \rangle_i, \forall n, \tag{15}$$

where

$$ind_l(c) \equiv ind_l(a * b) \text{ and } ind_u(c) \equiv ind_u(a * b). \tag{16}$$

In short, Eq. (15) can be written as $\langle c \rangle = \langle a \rangle * \langle b \rangle$. The relationships of the starting and terminal indices for the resulting sequence from the convolution of two sequences can be established in the following lemma.

Lemma 1: Associativity properties of starting and terminal indices are described as follows [30]:
(*i*) For $\langle a \rangle, \langle b \rangle \in \mathcal{S_C}$, $ind_l(a * b) = ind_l(a) + ind_l(b)$.
(*ii*) If $\langle a \rangle, \langle b \rangle \in \mathcal{S_C}$ are finite-support sequences, then $ind_u(a * b) = ind_u(a) + ind_u(b)$.
Proof: The proof is in Appendix A.

In addition, we provide the following lemma for the L_1-norms of the sequences, which can be adopted to quantify the approximation errors later on.

Lemma 2: There are K sequences $\langle a_1 \rangle, \langle a_2 \rangle, \cdots, \langle a_K \rangle \in \mathcal{S_C}$ and $K \in \mathbb{N}$, where $\|a_k\|_1 \equiv \sum_{n=-\infty}^{\infty} |\langle a_k \rangle_n| < \infty$

(absolutely-summable), for $k = 1, 2, \cdots, K$ (Note that the symbolic notation a_k inside $\langle\ \rangle$ is just the *sequence handle* and does not represent any value). We can obtain

$$\|a_1 * a_2 * \cdots * a_K\|_1 \leq \|a_1\|_1 \|a_2\|_1 \cdots \|a_K\|_1, \quad (17)$$

where

$$\langle c \rangle \equiv \langle a_1 \rangle * \langle a_2 \rangle * \cdots * \langle a_K \rangle \quad (18)$$

and

$$\|c\|_1 \equiv \|a_1 * a_2 * \cdots * a_K\|_1 = \sum_{n=-\infty}^{\infty} |\langle c \rangle_n|. \quad (19)$$

Moreover, if $\langle a_1 \rangle \geq 0, \langle a_2 \rangle \geq 0, \cdots, \langle a_K \rangle \geq 0$, then the equality in Eq. (17) holds.

Proof: The proof is in Appendix A.

Given two sequences $\langle a \rangle$ and $\langle b \rangle$, the addition (subtraction) operation of these two sequence, denoted by $\pm$, results in another sequence $\langle c \rangle$ such that

$$\langle c \rangle \equiv \langle a \rangle \pm \langle b \rangle, \quad (20)$$

where $\langle c \rangle_n \equiv \langle a \rangle_n \pm \langle b \rangle_n, \forall n$. According to the following theorem, $(\mathcal{S_C}, +, *)$ forms a field.

Theorem 1: The set of all causal sequences, $\mathcal{S_C}$, together with two operations, namely addition $+$ and convolution $*$, form a field $(\mathcal{S_C}, +, *)$. The addition identity is a zero sequence, i.e., $\langle \mathbf{0} \rangle \equiv (\cdots, 0, 0, \overline{0}, 0, \cdots)$ and the multiplication identity is denoted as $\langle \mathbf{1} \rangle \equiv (\cdots, 0, \overline{1}, 0, \cdots)$ where the overline specifies the 0-th indexed position of a sequence [30].

Proof: The proof is in Appendix A.

Let $\widetilde{M}$ be the function set of $\Big\{ f : \exists \langle a \rangle \in \mathcal{S_C}$ such that $f(z) = \sum_{k=-\infty}^{\infty} \langle a \rangle_k z^{-k}$ for all $z \in U^*_{R_a}(0) \Big\}$, where $U^*_{R_a}(0)$ is the punctured disk (centered at 0) for the convergence region of $\langle a \rangle$ with a convergence radius R_a.

Lemma 3: $f(z) \in \widetilde{M}$ if and only if there exists $k \in \mathbb{Z}$ such that the function $g(z) \equiv z^{-k} f(z)$ satisfies the following conditions [30]:

(A) there exists $\langle b \rangle \in \mathcal{S_C}$ such that $g(z) = \sum_{k=0}^{\infty} \langle b \rangle_k z^{-k}$ for all $z \in U_{R_b}(0)$, where $U_{R_b}(0)$ is the disk (centered at 0) for the convergence region of $\langle b \rangle$ with a convergence radius R_b;

(B) $g(0) = \langle b \rangle_0 \neq 0$.

Proof: The proof is in Appendix A.

According to Lemma 3, we can show that $\widetilde{M}$ forms a field.

Theorem 2: $(\widetilde{M}, +, \cdot)$ is a field [30].

Proof: The proof is in Appendix A.

The equivalence between causal sequences and z-functions is established by the following field isomorphism between the set of causal sequences and the set of z-functions.

Theorem 3: The Z-transform, $Z : \mathcal{S_C} \to \widetilde{\mathcal{M}}$, is a field isomorphism [30].

Proof: Obviously, $Z : \mathcal{S_C} \to \widetilde{\mathcal{M}}$ is linear. Moreover, if $\langle a \rangle \in \mathcal{S_C}$ with the radius of convergence R_a and $\langle b \rangle \in \mathcal{S_C}$ with the radius of convergence R_b, then the radius of convergence for $\langle c \rangle = \langle a * b \rangle$ is $R_c = \min\{R_a, R_b\}$. Assume that $\langle a \rangle_n = \alpha_n$, $\langle b \rangle_n = \beta_n$, and $\langle c \rangle_n = \zeta_n$. Hence,

$$\begin{aligned}
&\left| \left(\sum_{k=0}^{n} \alpha_k z^{-k} \right) \left(\sum_{k=0}^{n} \beta_k z^{-k} \right) - \sum_{k=0}^{\infty} \zeta_k z^{-k} \right| \\
&= \big| (\alpha_0 + \alpha_1 z^{-1} + \cdots + \alpha_n z^{-n}) \cdot \\
&\quad (\beta_0 + \beta_1 z^{-1} + \cdots + \beta_n z^{-n}) \\
&\quad - [\alpha_0\beta_0 + (\alpha_1\beta_0 + \alpha_0\beta_1) z^{-1} + \cdots] \big| \\
&= \big| (\alpha_0\beta_0 + (\alpha_1\beta_0 + \alpha_0\beta_1) z^{-1} + \cdots \\
&\quad + (\alpha_0\beta_n + \cdots + \alpha_n\beta_0) z^{-n} \\
&\quad + (\alpha_1\beta_n + \alpha_2\beta_{n-1} + \cdots + \alpha_n\beta_1) z^{-(n+1)} \\
&\quad + \cdots + \alpha_n\beta_n z^{-2n}) \\
&\quad - [\alpha_0\beta_0 + (\alpha_1\beta_0 + \alpha_0\beta_1) z^{-1} + \cdots] \big| \\
&= \left| \sum_{k=n+1}^{2n} \widetilde{\zeta}_k z^{-k} + \sum_{k=2n+1}^{\infty} \zeta_k z^{-k} \right| \\
&\leq \sum_{k=n+1}^{\infty} \left(\sum_{j=0}^{k} |\alpha_{k-j}||\beta_j| \right) |z^{-1}|^k,
\end{aligned}$$

where $\widetilde{\zeta}_k = \sum \alpha_{k-j}\beta_j$ and $j \geq (n+1)$.

Since

$$\begin{aligned}
&\sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} |\alpha_{k-j}||\beta_j| \right) |z^{-1}|^k \\
&\leq \left(\sum_{n=0}^{\infty} |\alpha_n||z^{-1}|^n \right) \left(\sum_{n=0}^{\infty} |\beta_n||z^{-1}|^n \right) < \infty,
\end{aligned}$$

from Lemma 2, we have $\sum_{k=n+1}^{\infty} \left(\sum_{j=0}^{k} |\alpha_{k-j}||\beta_j| \right) |z^{-1}|^k < \epsilon$ when n is large for any $\epsilon > 0$. This concludes that $Z(a) \cdot Z(b) = Z(c) = Z(a * b)$ and $Z : \mathcal{S_C} \to \widetilde{\mathcal{M}}$ is both linear and multiplicative.

If $Z(a) = f(z) = 0$, then $g(z) = z^{-k} f(z) = 0$ for $z = 0$ since $g(z)$ is a continuous function on $z \in U_{R_a}(0)$. Consequently, $g(0) = \alpha_k = 0$, which implies that $\langle a \rangle = \langle \mathbf{0} \rangle$. Thus, Z-transform is a one-to-one and onto (from the definition of $\widetilde{M}$) mapping. Because Z-transform is a linear and multiplicative bijection map from $\mathcal{S_C}$ to the field $\widetilde{M}$, we conclude that $\mathcal{S_C}$ is a field. □

IV. Error Bounds for Finite-Support Approximations of Inverse Mixing-Phase Filters

We have proved that the Z-transform has the property of field isomorphism. In this section, we can apply such a mathematical paradigm to derive the error bounds for the stabilized finite-support approximates of the inverse mixing-phase filters. As previously discussed in Section II, the exact L_p approximation error in Eq. (11) is very complicated to calculate. In this section, we will derive the L_1/L_2 error-bounds using the analysis we establish in Section III alternatively.

Since the focused inverse system (filter) approximation is finite-support, we use a finite-support sequence $\langle a \rangle \in \mathcal{S_F}$ to represent a finite-length filter and $\langle a \rangle =$

$(\cdots, 0, \alpha_0, \alpha_1, \cdots, \alpha_L, 0, \cdots)$, where $\langle a \rangle_0 = \alpha_0$ and $L \in \mathbb{N}$. Without loss of generality, we may assume that $\alpha_L = 1$. The corresponding (classical) Z-transform of $\langle a \rangle$ is $Z(a) = \alpha_0 + \alpha_1 z^{-1} + \alpha_2 z^{-2} + \cdots + z^{-L}$. Thus, $Z(a)$ can be factorized as

$$Z(a) = (z^{-1} - B_1)^{N_1}(z^{-1} - B_2)^{N_2} \cdots (z^{-1} - B_j)^{N_j} \cdot (z^{-1} - b_1)^{n_1}(z^{-1} - n_2)^{n_2} \cdots (z^{-1} - b_k)^{n_k}, \quad (21)$$

where $|B_l| > 1$ for $1 \le l \le j$ and $|b_l| < 1$ for $1 \le l \le k$ ($Z(a)$ cannot contain any zero on the unit circle as previously discussed). Note that $N_1 + N_2 + \cdots + N_j + n_1 + n_2 + \cdots + n_k = L$. Computing $Z^{-1}(a) = Z(a^{-1})$ (according to the field isomorphism) and using the partial fractional decomposition, we obtain

$$\begin{aligned} Z(a^{-1}) &= \frac{1}{(z^{-1} - B_1)^{N_1}(z^{-1} - B_2)^{N_2} \cdots (z^{-1} - B_j)^{N_j}} \\ &\quad \times \frac{1}{(z^{-1} - b_1)^{n_1}(z^{-1} - n_2)^{n_2} \cdots (z^{-1} - b_j)^{n_k}} \\ &= \left[\frac{\rho_{11}}{z^{-1} - B_1} + \cdots + \frac{\rho_{1N_1}}{(z^{-1} - B_1)^{N_1}}\right] \\ &\quad + \cdots + \left[\frac{\rho_{j1}}{z^{-1} - B_j} + \cdots + \frac{\rho_{jN_j}}{(z^{-1} - B_j)^{N_j}}\right] \\ &\quad + \left[\frac{\eta_{11}}{z^{-1} - b_1} + \cdots + \frac{\eta_{1n_1}}{(z^{-1} - b_1)^{n_1}}\right] \\ &\quad + \cdots + \left[\frac{\eta_{k1}}{z^{-1} - b_k} + \cdots + \frac{\eta_{kn_k}}{(z^{-1} - b_k)^{n_k}}\right], \end{aligned} \quad (22)$$

where $|B_j| > 1$ and $|b_j| < 1$, $\forall j$. From the above expansion, note that it is important to study the property of the inverse for the sequence $\langle a_b \rangle$ which is defined as $\langle a_b \rangle \equiv (\cdots, 0, \overline{-b}, 1, 0, \cdots)$. If $|b| > 1$, we use a causal sequence to invert $\langle a_b \rangle$. If $|b| < 1$, we use an anti-causal sequence to invert $\langle a_b \rangle$ instead. Define $\langle a_b^n \rangle$ as a sequence resulting from convolving $\langle a_b \rangle$ with itself $(n-1)$ times and $\langle a_b^{-n} \rangle$ as the inverse sequence of $\langle a_b^n \rangle$, respectively, where $\langle a_b^n \rangle \equiv \underbrace{\langle a_b \rangle * \langle a_b \rangle * \cdots * \langle a_b \rangle}_{(n-1) \text{ times convolution}}$ and $\langle a_b^n \rangle * \langle a_b^{-n} \rangle = \langle \mathbf{1} \rangle$. In order to evaluate the coefficients for an inverse sequence of $\langle a_b \rangle$ with multiplicity, we have the following lemma.

Lemma 4: Let $\langle a_b \rangle = (\cdots, 0, \overline{-b}, 1, 0, \cdots)$ for some $|b| > 1$. The j-th ($j \ge 0$) indexed element $\langle a_b^{-n} \rangle_j$ of $\langle a_b^{-n} \rangle$ is $\frac{(-1)^n \binom{j+n-1}{n-1}}{b^{n+j}}$ and $\|a_b^{-n}\|_1 = \|a_b^{-1}\|_1^n = \frac{1}{(|b|-1)^n}$. If $|b| < 1$, then the j-th ($j \le -n$) indexed element of $\langle a_b^{-n} \rangle$ is $\binom{j-1}{n-1} b^{-j-n}$ and $\|a_b^{-n}\|_1 = \|a_b^{-1}\|_1^n = (1 - |b|)^n$.

Proof: The proof is in Appendix B.

For the future approximation-error-bound derivation, we will use the following combinatorial identity in [27].

Lemma 5: Given two positive integers p and q, we have

$$\begin{aligned} &\binom{p}{q-1} - \binom{p+1}{q-1}\binom{q}{1} + \binom{p+2}{q-1}\binom{q}{2} \\ &\quad + \cdots + (-1)^{q-1}\binom{p+q-1}{q-1}\binom{q}{q-1} \\ &= (-1)^{q-1}\binom{p+q}{q-1}, \end{aligned} \quad (23)$$

where $\binom{m}{n}$ will be set to 0 if $m < n$.

We derive the following lemma to characterize the relationship between the (classical) Z-transform of an FIR filter and its corresponding truncated inverse. From now on, for notational convenience, we declare that $\alpha_b^{-n}(k) \equiv \langle a_b^{-n} \rangle_k$ denotes the k-th indexed element of $\langle a_b^{-n} \rangle$.

Lemma 6: Let $|b| > 1$ and $\langle a_b^{-n}(M) \rangle$ be the *truncated sequence* of $\langle a_b^{-n} \rangle$ by preserving the 0-th to $(M-1)$-th indexed elements of $\langle a_b^{-n} \rangle$. Let $\langle C_b^n(M) \rangle$ be the sequence resulting from the convolution of $\langle a_b^n \rangle$ and $\langle a_b^{-n}(M) \rangle$. If $M \ge (n+1)$, then the Z-transform of $\langle C_b^n(M) \rangle$ can be formulated as

$$\begin{aligned} Z(C_b^n(M)) &= 1 + \sum_{k=0}^{n-1} \alpha_b^{-n}(M-k-1) \times \\ &\quad \left(\sum_{l=0}^{n-k-1} \binom{n}{l} z^{-(n-l)}(-b)^l\right) z^{-(M-k-1)}. \end{aligned} \quad (24)$$

Proof: The proof is in Appendix B.

Eq. (24) can be adopted to derive the L_1 approximation error bound for the stabilized causal inverse of any FIR filter (system). In order to achieve this goal, here we provide a useful lemma:

Lemma 7: When $|b| > 1$, the L_1 error between $\langle C_b^n(M) \rangle$ and the identity sequence $\langle \mathbf{1} \rangle = (\cdots, 0, \overline{1}, 0, \cdots)$, can be bounded as

$$\begin{aligned} \|C_b^n(M) - \mathbf{1}\|_1 &\le \sum_{i=1}^{n} \frac{\binom{n+M-i-1}{n-1} \sum_{k=0}^{n-i} \binom{n}{k} |b|^k}{|b|^{n+M-i}} \\ &\equiv E_>^n(b, M). \end{aligned} \quad (25)$$

Proof: The proof is in Appendix B.

For the case of $|b| < 1$, we address another lemma similar to Lemma 6 as below.

Lemma 8: Let $|b| < 1$ and $\langle a_b^{-n}(N) \rangle$ be the corresponding truncated sequence of $\langle a_b^{-n} \rangle$ by preserving the (-1)-th to $(-N)$-th indexed elements. In addition, let $\langle C_b^n(N) \rangle$ denote the convolution of $\langle a_b^n \rangle$ and $\langle a_b^{-n}(N) \rangle$. If $N \ge (n+1)$, then the Z-transform of $\langle a_b^{-n}(N) \rangle$ is

$$\begin{aligned} Z(C_b^n(N)) &= 1 + \sum_{k=0}^{n-1} \alpha_b^{-n}(-N+k) \times \\ &\quad \left[\sum_{l=0}^{n-k-1} \binom{n}{l} z^{-(n-l)}(-b)^l\right] z^{N-k}, \end{aligned} \quad (26)$$

where $\alpha_b^{-n}(-N+k)$ denotes the $(-N+k)$-th indexed element of $\langle a_b^{-n} \rangle$, i.e., $\alpha_b^{-n}(-N+k) \equiv \langle a_b^{-n} \rangle_{(-N+k)}$.

Proof: The proof is in Appendix B.

As a result, we can derive the following formula of the L_1 approximation error bound which is similar to the lemma 7 for $|b| < 1$ as well.

Lemma 9: When $|b| < 1$, the L_1 error between $\langle C_b^n(N) \rangle$ and the identity sequence $\langle \mathbf{1} \rangle$ can be bounded as

$$\begin{aligned} &\|C_b^n(N) - \mathbf{1}\|_1 \\ &\quad \le \sum_{i=1}^{n} \left[\binom{N-i}{n-1} \sum_{k=0}^{n-i} \binom{n}{k} |b|^k\right] |b|^{N-n-i+1} \\ &\quad \equiv E_<^n(b, N). \end{aligned} \quad (27)$$

Proof: The proof is in Appendix B.

Using the error bounds in Eqs. (25) and (27), we derive the following theorem, which will provide the L_1 approximation error bound associated with a finite-support sequence approximate, e.g., $\langle a^{-1}\rangle_{i=-N}^{M-1} \in \mathcal{S}_{\mathcal{F}}$, for the inverse of an FIR system represented as $\langle a\rangle \in \mathcal{S}_{\mathcal{F}}$.

Theorem 4: Let $\langle a\rangle \in \mathcal{S}_{\mathcal{F}}$ and it can be written as $\langle a\rangle = (\cdots, 0, \overline{\alpha_0}, \alpha_1, \cdots, \alpha_L, 0, \cdots)$. Without loss of generality, we may assume that $a_L = 1$. Given two positive integers M and N, the Z-transform of the inverse filter with the only nontrivial elements indexed from $-N$ to $M-1$ (truncating the exact inverse filter to a length-$(N+M)$ finite-support filter $\langle a^{-1}(M,N)\rangle$) can be approximated as

$$\begin{aligned} Z(a^{-1}) \;\approx\; & \sum_{p=1}^{j}\sum_{q=1}^{N_p} \rho_{pq} Z(C_{B_p}^{q}(M)) \\ & + \sum_{p=1}^{k}\sum_{q=1}^{n_p} \eta_{pq} Z(C_{b_p}^{q}(N)) \\ \equiv\; & Z(a^{-1}(M,N)), \end{aligned} \tag{28}$$

where $\rho_{pq}, B_p, N_p, \eta_{pq}, b_p$ and n_p can be determined according to Eq. (22). Here we derive the L_1 error between such a sequence $\langle a^{-1}(M,N)\rangle$ and the sequence $\langle \mathbf{1}\rangle$ as

$$\begin{aligned} \xi_1(a, a^{-1}(M,N)) &= \|a * a^{-1}(M,N) - \mathbf{1}\|_1 \\ &\le \sum_{p=1}^{j}\sum_{q=1}^{N_p} |\rho_{pq}| E_{>}^{q}(B_p, M) \widehat{\frac{Z(a)}{(z^{-1}-B_p)^q}} \\ &\quad + \sum_{p=1}^{k}\sum_{q=1}^{n_p} |\eta_{pq}| E_{<}^{q}(b_p, N) \widehat{\frac{Z(a)}{(z^{-1}-b_p)^q}}, \end{aligned} \tag{29}$$

where $\widehat{\frac{Z(a)}{(z^{-1}-B_p)^q}}$ and $\widehat{\frac{Z(a)}{(z^{-1}-b_p)^q}}$ represent the summations of the absolute values for all coefficients associated with the polynomial expansions of $\frac{Z(a)}{(z^{-1}-B_p)^q}$ and $\frac{Z(a)}{(z^{-1}-b_p)^q}$, respectively [26]; $E_{>}^{q}(B_p, M), E_{<}^{q}(b_p, N)$ are defined in Lemmas 7 and 9, respectively.

Proof:

The Z-transform $Z(a^{-1}(M,N))$ of the sequence $\langle a^{-1}(M,N)\rangle$ can be determined according to Lemma 6 and Lemma 8. Thus, according to Lemma 7 and Lemma 9, the L_1 approximation error bound can be expressed as

$$\begin{aligned} &\|a * a^{-1}(M,N) - \mathbf{1}\|_1 \\ &= \left\| Z^{-1}\left(Z(a) Z(a^{-1}(M,N)) - 1 \right) \right\|_1 \\ &= \Bigg\| Z^{-1}\Bigg(\sum_{p=1}^{j}\sum_{q=1}^{N_p} \frac{\rho_{pq} Z(a)}{(z^{-1}-B_p)^q} \left[1 + Z(C_{B_p}^{q}(M)) - 1\right] \\ &\quad + \sum_{p=1}^{j}\sum_{q=1}^{n_p} \frac{\eta_{pq} Z(a)}{(z^{-1}-b_p)^q} \left[1 + Z(C_{b_p}^{q}(N)) - 1\right] - 1 \Bigg) \Bigg\|_1 \\ &= \Bigg\| Z^{-1}\Bigg(\sum_{p=1}^{j}\sum_{q=1}^{N_p} \frac{\rho_{pq} Z(a)}{(z^{-1}-B_p)^q} (Z(C_{B_p}^{q}(M)) - 1) \\ &\quad + \sum_{p=1}^{j}\sum_{q=1}^{n_p} \frac{\eta_{pq} Z(a)}{(z^{-1}-b_p)^q} (Z(C_{b_p}^{q}(N)) - 1) \Bigg) \Bigg\|_1 \\ &\le \sum_{p=1}^{j}\sum_{q=1}^{N_p} |\rho_{pq}| E_{>}^{q}(B_p, M) \widehat{\frac{Z(a)}{(z^{-1}-B_p)^q}} \\ &\quad + \sum_{p=1}^{k}\sum_{q=1}^{n_p} |\eta_{pq}| E_{<}^{q}(b_p, N) \widehat{\frac{Z(a)}{(z^{-1}-b_p)^q}}. \end{aligned} \tag{30}$$

□

In addition, we can also derive the L_2 approximation error bound as follows.

Lemma 10: Let $\langle a\rangle \in \mathcal{S}_{\mathcal{F}}$ and its truncated finite-support inverse filter corresponds to $a^{-1}(M,N)$ according to Theorem 4. Then the L_2 approximation error between $\langle a * a^{-1}(M,N)\rangle$ and $\langle \mathbf{1}\rangle$ is bounded as

$$\begin{aligned} &\xi_2(a, a^{-1}(M,N)) = \left\| a * a^{-1}(M,N) - \mathbf{1} \right\|_2^2 \\ &\le \Bigg[\sum_{p=1}^{j}\sum_{q=1}^{N_p} |\rho_{pq}| E_{>}^{q}(B_p, M) \widehat{\frac{Z(a)}{(z^{-1}-B_p)^q}} \\ &\quad + \sum_{p=1}^{k}\sum_{q=1}^{n_p} |\eta_{pq}| E_{<}^{q}(b_p, N) \widehat{\frac{Z(a)}{(z^{-1}-b_p)^q}} \Bigg]^2, \end{aligned} \tag{31}$$

where all the parameters are defined in Eqs. (28) and (29).

Proof: The L_2 norm of any sequence $\langle a\rangle$ can be expressed as

$$\|a\|_2^2 \equiv \sum_n |\langle a\rangle_n|^2.$$

Hence, for any sequence $\langle a\rangle$, we have $\|a\|_2^2 \le \|a\|_1^2$. Thus,

$$\left\| a * a^{-1}(M,N) - \mathbf{1} \right\|_2^2 \le \left\| a * a^{-1}(M,N) - \mathbf{1} \right\|_1^2. \tag{32}$$

According to Eqs. (29) and (32), Lemma 10 holds. □

We will provide an example here to show that the L_1-bound derived in Theorem 4 is achievable. Although the L_2-bound derived in Lemma 10 is quite loose in general, Lemma 10 facilitates an upper bound for the mean-square approximation error based on our derived L_1-bound.

Example 1: Given a sequence $\langle a\rangle \in \mathcal{S}_{\mathcal{F}}$ with the transfer function $Z(a) = 1 - \frac{7}{3}z^{-1} + \frac{2}{3}z^{-2}$, we can calculate the L_1-error bound of $\| \ a * a^{-1}(2,3) - \mathbf{1} \ \|_1 = \frac{19}{45}$ via some algebraic manipulations. From Lemma 7 and Lemma 9, we get $E_{>}^{1}(2,2) = \frac{1}{4}$ and $E_{<}^{1}(\frac{1}{3},3) = \frac{1}{27}$. Since $\frac{1}{Z(a)} = \frac{6/5}{1-2z^{-1}} + \frac{-1/5}{1-1/3z^{-1}}$, the right-hand side of Eq. (29) is equal to $\frac{1}{4}\cdot\frac{6}{5}\cdot\frac{4}{3} + \frac{1}{27}\cdot\frac{1}{5}\cdot 3$. This value is exactly the same as $\frac{19}{45}$. Hence, the L_1-error bound is the same as the exact L_1 approximation error in this example. In other words, this bound is achievable.

V. An Application for Communication Systems

In this section, we will illustrate how to apply our derived theoretical analysis in previous sections for the equalizer design subject to the error constraint and also demonstrate the corresponding numerical evaluation.

A. Error Bound Application for Equalizers in Telecommunications

Our newly derived error bound in Section IV can be applied for a wide variety of applications involving the inverse system approximation using the finite impulse responses. In this section, we will discuss a typical example of such applications, namely the equalizer design for telecommunication receivers. We will design an equalizer using an FIR filter, whose impulse response function can be expressed as a sequence $\langle h \rangle = (\cdots, 0, h[0], h[1], \cdots, h[L], 0, \cdots)$, where $\langle h \rangle_n = h[n]$. Then $H(z) = Z(h)$ is the corresponding Z-transform. That is

$$\begin{aligned} H(z) &= \sum_{n=0}^{L} h[n] z^{-n} \\ &= h[0] + h[1]z^{-1} + \cdots + h[L]z^{-L}. \end{aligned} \tag{33}$$

Applying the spectral factorization, we obtain

$$H(z) = H_{min}(z) H_{max}(z), \tag{34}$$

where

$$H_{max}(z) = \prod_{i=1}^{P} (1 - B_i z^{-1})^{N_i}, |B_i| > 1, \forall i, \tag{35}$$

$$H_{min}(z) = \prod_{i=1}^{Q} (1 - b_i z^{-1})^{n_i}, |b_i| < 1, \forall i, \tag{36}$$

and B_i, b_i are the zeros of $H(z)$.

The objective for designing an equalizer is to determine an inverse system $H^{-1}(z)$ such that $H^{-1}(z)H(z) = 1$. However, very often, $H^{-1}(z)$ can not be realized as a finite-support or causal filter. For practical signal processing applications, we may just approximate this inverse system as $H^{-1}(z) \approx G_{max}(z) H_{min}^{-1}(z) = G_{min}(z) G_{max}(z)$ with

$$G_{max}(z) = z^{-N} \sum_{i=1}^{P} \sum_{j=1}^{N_i} \delta_{ij} Z\left(C^{j}_{\frac{-1}{B_i}}(N) \right) \tag{37}$$

and

$$G_{min}(z) = \frac{1}{H_{min}(z)}, \tag{38}$$

where δ_{ij} are determined from the partial fractional decomposition of $\frac{1}{H_{max}(z)}$ and $Z\left(C^{j}_{\frac{-1}{B_i}}(N) \right)$ arises from Lemma 8 since $|\frac{-1}{B_i}| < 1$. Note that the truncation parameter N corresponds to the delay value $\triangle$ defined in Eq. (3) for stabilizing the inverse approximate and making it causal. This proposed inverse system or equalizer $G_{min}(z)G_{max}(z)$ is plotted in Fig. 1, where $x[n]$, $r[n]$, $H(z)$, $\hat{x}[n]$ denote the transmitted signal, the received signal, the channel transfer function and the equalized signal, respectively. Note that we do not consider the *equalized noise power* here for any additive white channel noise since we can easily normalize the equalizer's impulse response by a factor to achieve a unity L_2-norm such that the equalized noise power is always an identical constant for any FIR equalizer [31].

B. Numerical Evaluation

Here we will present the numerical results of an example for the aforementioned inverse filter design. In Figures 2 and 3, we plot the derived L_1 error bounds given by Eq. (30) between the exact inverse systems and their stabilized approximates with respect to the truncation parameter M. In this particular example, there are only two zeros contained in the system transfer function $H(z)$, namely B_1 ($|B_1| > 1$) and b_1 ($|b_1| < 1$). In Fig. 2, different error bounds are depicted for different values of B_1 and b_1. It can be observed that the larger M the smaller the error bound because the less truncation takes place for approximating the inverse system. For a fixed M, when $|B_1|$ becomes large and $|b_1|$ becomes small, we find that the error bound diminishes. This phenomenon arises from that the calculated inverse filter coefficients related to B_1 have the form of $\frac{(-1)^n \binom{j+n-1}{n-1}}{|B_1|^{n+j}}$ where n is the multiplicity of B_1 and j denotes the index of the inverse filter coefficients, according to Eq. (50). Therefore, when $|B_1|$ becomes large, each coefficient of the inverse filter becomes small in magnitude. On the other hand, the calculated inverse filter coefficients related to b_1 have the form of $\binom{-j-1}{n-1}|b_1|^{-j-n}$ where n is the multiplicity of b_1 and j denotes the index of the inverse filter coefficients. Therefore, when $|b_1|$ becomes small, each coefficient of the inverse filter becomes small in magnitude. In Fig. 3, different error bounds are also plotted for different multiplicities N_1 associated with $B_1 = 2$ and multiplicities n_1 associated with $b_1 = 1/2$. It is obvious that the L_1 error bound increases with the increasing multiplicities since the larger the multiplicities the larger the absolute value of each inverse filter coefficient.

In the following two figures, for telecommunication applications, we illustrate the effect of the truncation parameter M (the model order of the approximated inverse system) on the symbol-error probability P_e, where the channel transfer function $H(z)$ has the two zeros B_1 ($|B_1| > 1$) and b_1 ($|b_1| < 1$). The symbol-error probability is evaluated using Eq. (4) in [32] under an assumption of AWGN channel and BPSK modulation. The notions "ML" and "M" in Figures 4 and 5 are used to represent the memoryless and memory channel conditions, respectively. For a memoryless channel, the truncation parameter M does not affect the symbol-error probability because the inverse filter impulse response is a Dirac-delta sequence. In Fig. 4, we depict P_e versus M for different pairs of (B_1, b_1) and the signal-to-noise ratio (SNR) is set to be 3dB. When $|B_1|$ becomes large and $|b_1|$ becomes small, P_e decreases and approaches to the optimistic value for a memoryless channel. In Fig. 5, we plot P_e versus M for different channel SNRs when the channel transfer function is $(-2+z^{-1})(-1/2+z^{-1})$. Fig. 5 shows that P_e decreases with the increasing channel SNR.

VI. Conclusion

In this paper, we discuss and analyze the significant and fundamental problem of the finite-support approximation of any mixing-phase FIR filter (system). We employ a rigorous mathematical paradigm from the abstract algebra to characterize Z-transform. Besides, we establish the corresponding field isomorphism and maneuver the sequences algebraically thereupon to derive the L_1 and L_2 error bounds between an exact inverse filter with infinite coefficients and its truncated finite-support filter. We characterize such a truncation using the truncation parameter, which also corresponds to the model order of the inverse approximate. The monotonically increasing property of the L_1 approximation error bound with respect to the decreasing truncation parameter can be discovered in our new analysis. Our newly derived error bound can be applied to determine the appropriate model order or stabilizing delay for the inverse system approximation in numerous system applications of signal processing, control and communications.

APPENDIX

A. Proofs for the Isomorphism between Causal Sequences and z-functions

Here we will provide the detailed proofs for those lemmas and theories presented in Section III.

Proof of Lemma 1:

For (i), if $\langle c\rangle = \langle a\rangle * \langle b\rangle$, then $\langle c\rangle_k = \sum_{i=ind_l(b)}^{\infty} \langle a\rangle_{k-i}\langle b\rangle_i$. Thus, $\langle c\rangle_k = 0$, for $k < ind_l(a) + ind_l(b)$ and $\langle c\rangle_k \neq 0$, for $k = ind_l(a) + ind_l(b)$. Consequently, $ind_l(c) = ind_l(a * b) = ind_l(a) + ind_l(b)$.
For (ii), let $\langle a\rangle$ be a causal sequence with $ind_l(a) = k_a$ and $ind_u(a) = k_a + n'$ for some $n' \in \mathbb{N}$. Let $\langle b\rangle$ be another causal sequence with $ind_l(b) = k_b$ and $ind_u(b) = k_b + m'$, for some $m' \in \mathbb{N}$. Without loss of generality, we may assume that $k_a + n' \leq k_b + m'$. Thus, $\langle c\rangle_k = \sum_{i=k-k_a-n'}^{k_b+m'} \langle a\rangle_{k-i}\langle b\rangle_i$. Therefore, $\langle c\rangle_k = 0$ for $k - k_a - n' > k_b + m'$ or equivalently $k > k_a + n' + k_b + m'$. Note that $\langle c\rangle_k = \langle a\rangle_{k_a+n'}\langle b\rangle_{k_b+m'} \neq 0$ when $k = k_a + n' + k_b + m'$. Hence, $ind_u(a*b) = k_a + n' + k_b + m' = ind_u(a) + ind_u(b)$.
□

Proof of Lemma 2:

By induction, for $K = 2, 3, \ldots$, we can prove this lemma. For $K = 2$, we have $\|a_1 * a_2\|_1 = \sum_{k=-\infty}^{\infty} |\sum_{i=-\infty}^{\infty} \langle a_1\rangle_i\langle a_2\rangle_{k-i}| \leq \sum_{i=-\infty}^{\infty} |\langle a_1\rangle_i| \sum_{k=-\infty}^{\infty} |\langle a_2\rangle_{k-i}| = \|a_1\|_1\|a_2\|_1$. Therefore, the inequality holds for $K = 2$. If $\langle a_1\rangle, \langle a_2\rangle \geq 0$, we define the two normalized sequences $\langle\widetilde{a}_1\rangle, \langle\widetilde{a}_2\rangle$, whose elements can be specified as $\langle\widetilde{a}_1\rangle_n \equiv \frac{\langle a_1\rangle_n}{\|a_1\|_1}, \langle\widetilde{a}_2\rangle_n \equiv \frac{\langle a_2\rangle_n}{\|a_2\|_1}$. It yields

$$
\begin{aligned}
\|\widetilde{a}_1 * \widetilde{a}_2\|_1 &= \sum_{k=-\infty}^{\infty} (\sum_{i=-\infty}^{\infty} \langle\widetilde{a}_1\rangle_i\langle\widetilde{a}_2\rangle_{k-i}) \\
&= \sum_{k=-\infty}^{\infty} (\sum_{i=-\infty}^{\infty} \langle\widetilde{a}_1\rangle_{k-i}\langle\widetilde{a}_2\rangle_i) \\
&= \sum_{i=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \langle\widetilde{a}_1\rangle_{k-i}\langle\widetilde{a}_2\rangle_i \\
&= \sum_{i=-\infty}^{\infty} \langle\widetilde{a}_2\rangle_i \sum_{k=-\infty}^{\infty} \langle\widetilde{a}_1\rangle_{k-i} = 1.
\end{aligned}
$$

Hence, $\|\widetilde{a}_1 * \widetilde{a}_2\|_1 = \|\widetilde{a}_1\|_1\|\widetilde{a}_2\|_1$ and we obtain $\|a_1 * a_2\|_1 = \|a_1\|_1\|a_2\|_1$.

We assume that *Lemma 2* holds for $K = N$ for some $N \in \mathbb{N}$ and claim that it also holds for $K = N + 1$. Let's denote $\langle b\rangle \equiv \langle a_1\rangle * \langle a_2\rangle * \cdots * \langle a_N\rangle$ and the corresponding L_1-norm $\|b\|_1 < \infty$. This claim can be justified since the following can be easily derived:

$$
\begin{aligned}
\|a_1 * a_2 * \cdots * a_N * a_{N+1}\|_1 &= \|b * a_{N+1}\|_1 \\
&\leq \|b\|_1\|a_{N+1}\|_1 \\
&\leq \|a_1\|_1\|a_2\|_1 \cdots \|a_N\|_1\|a_{N+1}\|_1,
\end{aligned} \tag{39}
$$

and for $\langle a_1\rangle \geq 0, \langle a_2\rangle \geq 0, \cdots, \langle a_N\rangle \geq 0, \langle a_{N+1}\rangle \geq 0$,

$$
\begin{aligned}
\|a_1 * a_2 * \cdots * a_N * a_{N+1}\|_1 &= \|b * a_{N+1}\|_1 \\
&= \|b\|_1\|a_{N+1}\|_1 \\
&= \|a_1\|_1\|a_2\|_1 \cdots \|a_N\|_1\|a_{N+1}\|_1.
\end{aligned} \tag{40}
$$

Thus, *Lemma 2* is proved by mathematical induction. □

Proof of Theorem 1:

It is easy to verify that $(\mathcal{S}_\mathcal{C}, +)$ is an Abeliean group with the addition identity $\langle\mathbf{0}\rangle$. In order to show that $(\mathcal{S}_\mathcal{C}, +, *)$ is a field, we need to prove that $(\mathcal{S}_\mathcal{C}, +, *)$ is a commutative ring with multiplicative inverse. For notional convenience, we can further manipulate the sequence handles a, b, to simply denote the convolutional sequence in Eq. (15) as

$$\langle c\rangle = \langle a\rangle * \langle b\rangle \equiv \langle a * b\rangle, \tag{41}$$

and denote the additive (differential) sequence in Eq. (20) as

$$\langle c\rangle = \langle a\rangle \pm \langle b\rangle \equiv \langle a \pm b\rangle, \tag{42}$$

respectively. Consequently the commutativity of convolution can be established as follows:

$$\langle a * b\rangle = \langle b * a\rangle. \tag{43}$$

Similarly, the associativity of convolution can also be established as

$$
\begin{aligned}
\langle (a * b) * c\rangle &= \langle a * (c * b)\rangle \\
&= \langle a * (b * c)\rangle,
\end{aligned} \tag{44}
$$

while the distributivity of convolution can be obtained as

$$\langle a * (b + c)\rangle = \langle (a * b) + (a * c)\rangle. \tag{45}$$

Therefore, $(\mathcal{S}_\mathcal{C}, +, *)$ is a commutative ring. Next we need to show the existence of the multiplicative inverse of any causal sequence $\langle a\rangle \in \mathcal{S}_\mathcal{C}$ with $ind_l(a) = k$. According to Lemma 1, we can derive the inverse of $\langle a\rangle$ as

$$\langle a_I\rangle = (\cdots, 0, \beta_{-k}, \beta_{-k+1}, \beta_{-k+2}, \cdots). \tag{46}$$

Since $\langle\mathbf{1}\rangle = \langle a\rangle * \langle a_I\rangle = (\cdots, 0, \alpha_k\beta_{-k}, \alpha_{k+1}\beta_{-k} + \alpha_k\beta_{-k+1}, \alpha_{k+2}\beta_{-k} + \alpha_{k+1}\beta_{-k+1} + \alpha_k\beta_{-k+2}, \cdots)$, where $\langle a\rangle_k \equiv \alpha_k$, we require

$$
\begin{aligned}
\beta_{-k} &= \frac{1}{\alpha_k}, \\
\beta_{-k+1} &= -\frac{\alpha_{k+1}\beta_{-k}}{\alpha_k} = -\frac{\alpha_{k+1}}{\alpha_k^2}, \\
&\vdots \\
\beta_{-k+m} &= -\frac{\sum_{i=1}^{m} \alpha_{k-i+m+1}\beta_{-k+i-1}}{\alpha_k}, \forall m \in \mathbb{N}.
\end{aligned} \tag{47}
$$

This proves the existence of such an inverse $\langle a_I \rangle$. For the uniqueness, if there exist two inverse sequences, say $\langle a_I \rangle, \langle a'_I \rangle$ for $\langle a \rangle$ ($\langle a \rangle \neq \langle \mathbf{0} \rangle$), then $\langle a \rangle * \langle a_I \rangle - \langle a \rangle * \langle a'_I \rangle = \langle \mathbf{1} \rangle - \langle \mathbf{1} \rangle = \langle \mathbf{0} \rangle$. Hence, $\langle a \rangle * (\langle a_I \rangle - \langle a'_I \rangle) = \langle \mathbf{0} \rangle$. Because $\langle a \rangle \neq \langle \mathbf{0} \rangle$, we must have $\langle a_I \rangle - \langle a'_I \rangle = \langle \mathbf{0} \rangle$ and therefore $\langle a_I \rangle = \langle a'_I \rangle$.

Finally, since $(\mathcal{S}_{\mathcal{C}}, +, *)$ is a commutative ring with multiplicative inverse, $(\mathcal{S}_{\mathcal{C}}, +, *)$ forms a field. □

Proof of Lemma 3:

From the definition of $\widetilde{M}$, $f \in \widetilde{M}$ if and only if there exists $\langle a \rangle \in \mathcal{S}_{\mathcal{C}}$ with $ind_l(a) = k$ such that

$$\begin{aligned} f(z) &= \alpha_k z^{-k} + \alpha_{k+1} z^{-(k+1)} + \alpha_{k+2} z^{-(k+2)} + \cdots \\ &= z^{-k} g(z), \end{aligned}$$

where $\langle a \rangle_k \equiv \alpha_k$, $g(z) = (\alpha_k + \alpha_{k+1} z^{-1} + \alpha_{k+2} z^{-2} + \cdots)$ and $g(0) = \alpha_k \neq 0$.□

Proof of Theorem 2:

First, the two following nontrivial facts need to be proved:

(C) if $f_1, f_2 \in \widetilde{M}$, then $f_1 \cdot f_2 \in \widetilde{M}$;

(D) if $f \in \widetilde{M}$, then $\frac{1}{f} \in \widetilde{M}$.

For (C), let $g_1(z) = z^{-k_1} f_1(z)$ and $g_2(z) = z^{-k_2} f_2(z)$, where $g_1(z), g_2(z)$ satisfy the two conditions (A), (B) of Lemma 3. Then $(f_1 \cdot f_2)(z) = z^{k_1+k_2} g_1(z) g_2(z) = z^{k_1+k_2} g(z)$, where

$$\begin{aligned} g(z) = g_1(z) g_2(z) &= \left(\sum_{k=0}^{\infty} \langle b_1 \rangle_k z^{-k} \right) \left(\sum_{k=0}^{\infty} \langle b_2 \rangle_k z^{-k} \right) \\ &= \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} \langle b_1 \rangle_{k-i} \langle b_2 \rangle_i \right) z^{-k} = \sum_{k=0}^{\infty} \langle c \rangle_k z^{-k}. \end{aligned} \tag{48}$$

Note that $\langle c \rangle \equiv \langle b_1 * b_2 \rangle \in \mathcal{S}_{\mathcal{C}}$ and $g(0) = \langle c \rangle_0 = \langle b_1 \rangle_0 \langle b_2 \rangle_0 \neq 0$. Therefore, $f_1 \cdot f_2 \in \widetilde{M}$.

To prove (D), let $f(z) = z^k g(z)$, where $g(z)$ satisfies (A) and (B) of Lemma 3. Thus $\frac{1}{f(z)} = z^{-k} h(z)$, where $h(z) = \frac{1}{g(z)}$. Since $g(z)$ is analytic on $U_\epsilon(0)$ for some $\epsilon > 0$ and $g(0) \neq 0$, there exists $0 < \epsilon_1 \leq \epsilon$ such that $g(z) \neq 0$ for all $z \in U_{\epsilon_1}(0)$. Thereby, $h(z)$ is analytic on $U_{\epsilon_1}(0)$ and $h(0) \neq 0$. Since $\sum_{k=0}^{\infty} \langle b \rangle_k z^{-k}$ exists for some $z \neq 0$, then $\overline{\lim_{k \to \infty}} |\langle b \rangle_k|^{\frac{1}{k}} < \infty$, i.e., $\langle b \rangle \in \mathcal{S}_{\mathcal{C}}$. □

Definition 1: The Z-transform, $Z : \mathcal{S}_{\mathcal{C}} \to \widetilde{\mathcal{M}}$, is defined as

$$Z(a) \equiv f, \text{where } f(z) \equiv \sum_{k=-\infty}^{\infty} \langle a \rangle_k z^{-k}. \tag{49}$$

□

B. Proofs of Error Bounds for Finite-Support Inverse System Approximation

In this section, we will provide the detailed proofs for those lemmas presented in Section IV to derive the inverse system approximation-error bounds.

Proof of Lemma 4: Substituting z^{-1} with $\widetilde{z}$, we get $Z(a_b^n) = (z^{-1} - b)^n = (\widetilde{z} - b)^n$. Consequently,

$$\begin{aligned} Z(a_b^{-n}) &= Z^{-1}(a_b^n) = \frac{1}{(\widetilde{z} - b)^n} = \frac{(-1)^{n-1}}{(n-1)!} \frac{d^{n-1}}{d\widetilde{z}^{n-1}} \frac{1}{\widetilde{z} - b} \\ &= \frac{(-1)^{n-1}}{(n-1)!} \frac{d^{n-1}}{d\widetilde{z}^{n-1}} \sum_{k=0}^{\infty} \frac{-\widetilde{z}^k}{b^{k+1}} \\ &= \frac{(-1)^n}{(n-1)!} \sum_{k=0}^{\infty} \frac{k(k-1)\cdots(k-n+2)\widetilde{z}^{k-(n-1)}}{b^{k+1}} \\ &= \frac{(-1)^n}{b^n} \sum_{j=0}^{\infty} \frac{\binom{j+n-1}{n-1}}{b^j} \widetilde{z}^j \\ &= \frac{(-1)^n}{b^n} \sum_{j=0}^{\infty} \frac{\binom{j+n-1}{n-1}}{b^j} z^{-j}, \end{aligned}$$

and

$$\langle a_b^{-n} \rangle = \left(\cdots, 0, \overline{\frac{(-1)^n}{b^n}}, \frac{(-1)^n n}{b^{n+1}}, \frac{(-1)^n \binom{n}{2}}{b^{n+2}}, \cdots \right). \tag{50}$$

Thus, the j-th indexed element of $\langle a_b^{-n} \rangle$ is $\frac{(-1)^n \binom{j+n-1}{n-1}}{b^{n+j}}$.

For $|b| > 1$, we set $x = \frac{1}{|b|}$ and obtain

$$\|a_b^{-n}\|_1 = \frac{1}{|b|^n} \sum_{j=0}^{\infty} \frac{\binom{j+n-1}{n-1}}{|b|^j} = \frac{1}{x^n} \sum_{j=0}^{\infty} \frac{\binom{j+n-1}{n-1}}{x^j}. \tag{51}$$

Setting $k = j + n - 1$, we have

$$\begin{aligned} &\frac{1}{x^n} \sum_{j=0}^{\infty} \frac{\binom{j+n-1}{n-1}}{x^j} \\ &= \frac{1}{(n-1)!} \sum_{k=n-1}^{\infty} \frac{x^{k+1} k!}{(k-n+1)!} \\ &= \frac{1}{(n-1)!} \\ &\quad \times \sum_{k=n-1}^{\infty} k(k-1)(k-2)\cdots(k-(n-2)) x^{k+1} \\ &= \frac{1}{(n-1)!} \sum_{k=0}^{\infty} k(k-1)(k-2)\cdots(k-(n-2)) x^{k+1} \\ &= \frac{x^n}{(n-1)!} \\ &\quad \times \sum_{k=0}^{\infty} k(k-1)(k-2)\cdots(k-(n-2)) x^{k-(n-1)} \\ &= \frac{x^n}{(n-1)!} \frac{d^{n-1}}{dx^{n-1}} \sum_{k=0}^{\infty} x^k \\ &= \frac{x^n}{(n-1)!} \frac{d^{n-1}}{dx^{n-1}} \frac{1}{1-x} \\ &= \frac{x^n}{(1-x)^n} = \frac{1}{(|b|-1)^n}. \end{aligned} \tag{52}$$

According to Eqs. (51) and (52), $\|a_b^{-n}\|_1 = \frac{1}{(|b|-1)^n}$. Since

$\langle a_b^{-1}\rangle = (\cdots, 0, \frac{-1}{b}, \frac{-1}{b^2}, \cdots)$, we can show

$$\begin{aligned} \|a_b^{-1}\|_1^n &= \left[\sum_{j=1}^{\infty}\left(\frac{1}{|b|}\right)^j\right]^n \\ &= \left[\frac{1}{|b|}\frac{1}{\left(1-\frac{1}{|b|}\right)}\right]^n = \frac{1}{(|b|-1)^n}. \end{aligned} \quad (53)$$

On the other hand, for $|b| < 1$, we can express $Z(a_b^{-n})$ as

$$\begin{aligned} Z(a_b^{-n}) &= Z^{-1}(a_b^n) = \frac{1}{(\tilde{z}-b)^n} = \frac{(-1)^{n-1}}{(n-1)!}\frac{d^{n-1}}{d\tilde{z}^{n-1}}\frac{1}{\tilde{z}-b} \\ &= \frac{(-1)^{n-1}}{(n-1)!}\frac{d^{n-1}}{d\tilde{z}^{n-1}}\left(\sum_{k=0}^{\infty} b^k \tilde{z}^{-(k+1)}\right) \\ &= \frac{1}{(n-1)!}\sum_{k=0}^{\infty} b^k (k+1)\cdots(k+n-1)\tilde{z}^{-(k+n)} \\ &= \sum_{k=0}^{\infty} b^k \frac{(k+n-1)!}{k!(n-1)!}\tilde{z}^{-(k+n)} \\ &= \sum_{k=0}^{\infty} b^k \binom{k+n-1}{n-1}\tilde{z}^{-(k+n)} \\ &= \sum_{j=-n}^{-\infty} b^{-j-n}\binom{-j-1}{n-1} z^j. \end{aligned} \quad (54)$$

Hence,

$$\langle a_b^{-n}\rangle = \left(\cdots, b^3\binom{n+2}{n-1}, b^2\binom{n+1}{n-1}, b\binom{n}{n-1}, \overbrace{1}^{-n\text{-th position}}, 0, \cdots\right). \quad (55)$$

The j-th indexed element of $\langle a_b^{-n}\rangle$ is $b^{-j-n}\binom{-j-1}{n-1}$ for $j \leq -n$. The values of $\|a_b^{-n}\|_1$ and $\|a_b^{-1}\|_1^n$ are equal and they are equal to $(1-|b|)^n$ using the similar argument for the previous case of $|b| > 1$. □

Proof of Lemma 6:

Lemma 6 can be proved using the mathematical induction. First, we will prove that this lemma is valid for $M = (n+1)$. Inspecting the coefficients associated with z^{-j} in $Z(C_b^n(n+1))$, we carry out the following equation:

$$\begin{aligned} &\binom{j-1}{n-1} - \binom{j}{n-1}\binom{n}{1} + \cdots \\ &+ (-1)^{n-1}\binom{j+n-2}{n-1}\binom{n}{n-1} \\ &- (-1)^{n-1}\binom{j+n-1}{n-1} = 0. \end{aligned} \quad (56)$$

Eq. (56) immediately follows Lemma 5, for $1 \leq j \leq n$. Eq. (56) shows that the coefficients associated with z^{-j} in $Z(C_b^n(n+1))$ for $1 \leq j \leq n$ drop out, and therefore we only need to consider those coefficients associated with z^{-j} for $j > n$. Combining the associated terms with z^{-j} for $j > n$ we can express $Z(C_b^n(n+1))$ as

$$\begin{aligned} Z(C_b^n(n+1)) = 1 + \sum_{k=0}^{n-1}\alpha_b^{-n}(n-k) \times \\ \left(\sum_{l=0}^{n-k-1}\binom{n}{l} z^{-(n-l)}(-b)^l\right) z^{-(n-k)}, \end{aligned} \quad (57)$$

which is equivalent to Eq. (24) when $M = n+1$ is set.

Assuming that Eq. (24) is also valid for $M = s$, we have

$$\begin{aligned} Z(C_b^n(s)) = 1 + \sum_{k=0}^{n-1}\alpha_b^{-n}(s-k-1) \times \\ \left(\sum_{l=0}^{n-k-1}\binom{n}{l} z^{-(n-l)}(-b)^l\right) z^{-(s-k-1)}. \end{aligned} \quad (58)$$

Moreover, we claim that Eq. (24) is still valid for $M = (s+1)$. Furthermore, it is obvious that

$$\begin{aligned} &(-b+z^{-1})^n Z(C_b^n(s+1)) \\ &= (-b+z^{-1})^n\left[Z(C_b^n(s)) + \frac{(-1)^n}{b^{n+s}}\binom{n+s-1}{n-1} z^{-s}\right] \\ &= 1 + \sum_{k=0}^{n-1}\alpha_b^{-n}(s-k-1)\left[\sum_{l=0}^{n-k-1}\binom{n}{l} z^{-(n-l)}(-b)^l\right] \\ &\quad \times z^{-(s-k-1)} + (-b+z^{-1})^n\frac{(-1)^n}{b^{n+s}}\binom{n+s-1}{n-1} z^{-s} \\ &= 1 + \Big[\alpha_b^{-n}(s-n) + \alpha_b^{-n}(s-n+1)\binom{n}{1}(-b) \\ &\quad + \cdots + \alpha_b^{-n}(s-1)\binom{n}{n-1}(-b)^{n-1} \\ &\quad + \alpha_b^{-n}(s)(-b)^n\Big] z^{-s} \\ &\quad + \sum_{k=0}^{n-2}\alpha_b^{-n}(s-k-1) \times \\ &\quad \left[\sum_{l=0}^{n-k-2}\binom{n}{l} z^{-(n-l)}(-b)^l\right] z^{-(s-k-1)} \\ &\quad + \left[\sum_{l=0}^{n-1}\binom{n}{l} z^{-(n-l)}(-b)^l\right]\frac{(-1)^n}{b^{n+s}}\binom{n+s-1}{n-1} z^{-s} \\ &= 1 + \frac{(-1)^n}{b^{s+1}}\left[\binom{s}{n-1} - \binom{s+1}{n-1}\binom{n}{1}\right. \\ &\quad + \binom{s+2}{n-1}\binom{n}{2} + \cdots \\ &\quad + (-1)^{n-1}\binom{s+n-1}{n-1}\binom{n}{n-1} \\ &\quad \left.- (-1)^{n-1}\binom{s+n}{n-1}\binom{n}{n-1}\right] z^{-s} \end{aligned}$$

$$+\sum_{k=0}^{n-1}\alpha_b^{-n}((s+1)-k-1)\times \left[\sum_{l=0}^{n-k-1}\binom{n}{l}z^{-(n-l)}(-b)^l\right]z^{-((s+1)-k-1)}$$
$$=1+\sum_{k=0}^{n-1}\alpha_b^{-n}((s+1)-k-1)\times \left[\sum_{l=0}^{n-k-1}\binom{n}{l}z^{-(n-l)}(-b)^l\right]z^{-((s+1)-k-1)},$$

where the first four equalities are obtained by adding one more term to the sequence $C_b^n(s)$ and expanding this sequence $C_b^n(s)$ according to Lemma 4. From Lemma 5, the last equality is valid since the coefficient associated with z^{-s} is equal to 0. Consequently, Eq. (24) holds for $M = s+1$ and thereby such an identity is valid for all $M \geq (n+1)$ according to the mathematical induction. □

Proof of Lemma 7:

Since $Z(C_b^n(M)) - 1$ is equal to $\sum_{k=0}^{n-1}\alpha_b^{-n}(M-k-1)\left[\sum_{l=0}^{n-k-1}\binom{n}{l}z^{-(n-l)}(-b)^l\right]z^{-(M-k-1)}$, we have

$$\|C_b^n(M)-\mathbf{1}\|_1 \leq \sum_{k=0}^{n-1}|\alpha_b^{-n}(M-k-1)|\left[\left|\sum_{l=0}^{n-k-1}\binom{n}{l}z^{-(n-l)}(-b)^l\right|\right]$$
$$\leq \sum_{k=0}^{n-1}\frac{\binom{n+M-k-2}{n-1}}{|b|^{n+M-k-1}}\left[\left|\sum_{l=0}^{n-k-1}\binom{n}{l}|b|^l\right|\right]$$
$$\leq \sum_{i=1}^{n}\frac{\binom{n+M-i-1}{n-1}\sum_{k=0}^{n-i}\binom{n}{k}|b|^k}{|b|^{n+M-i}}.$$

□

Proof of Lemma 8:

Lemma 8 can also be justified using the mathematical induction. We will first prove that it is valid for $N = (n+1)$. Inspecting the coefficients associated with z^j in $Z(C_b^n(n+1))$, we obtain

$$\binom{j-1}{n-1}-\binom{j}{n-1}\binom{n}{1}+\cdots +(-1)^{n-1}\binom{j+n-2}{n-1}\binom{n}{n-1} -(-1)^{n-1}\binom{j+n-1}{n-1}=0 \quad . \tag{59}$$

Following Lemma 5, we conclude that Eq. (59) sustains for $1 \leq j \leq n$. Accordingly, the associated terms with z^j for $1 \leq j \leq n$ will be zeroed out, and only those terms associated with z^j for $j = (n+1)$ in $Z(C_b^n(n+1))$ remains. Hence, we can write $Z(C_b^n(n+1))$ as

$$Z(C_b^n(n+1)) = 1+\sum_{k=0}^{n-1}\alpha_b^{-n}(-n-1-k)\times \left[\sum_{l=0}^{n-k-1}\binom{n}{l}z^{-(n-l)}(-b)^l\right]z^{-(n+1-k)}, \tag{60}$$

which is equivalent to Eq. (26) for $N = n+1$.

Next, assuming that Eq. (26) is valid for $N = s$ for some $s \in \mathbb{N}$ and $s > (n+1)$, we have

$$Z(C_b^n(s)) = 1+\sum_{k=0}^{n-1}\alpha_b^{-n}(-s+k)\times \left[\sum_{l=0}^{n-k-1}\binom{n}{l}z^{-(n-l)}(-b)^l\right]z^{s-k}. \tag{61}$$

We further claim that Eq. (26) is also valid for $N = (s+1)$. Because $(-b+z^{-1})^n Z(C_b^n(s+1))$ is equal to $(-b+z^{-1})^n\left[Z(C_b^n(s))+\binom{s}{n-1}b^{s-n+1}z^{s+1}\right]$, we achieve

$$(-b+z^{-1})^n Z(C_b^n(s+1)) = (-b+z^{-1})^n\left[Z(C_b^n(s))+\binom{s}{n-1}b^{s-n+1}z^{s+1}\right]$$
$$=1+\sum_{k=0}^{n-1}\alpha_b^{-n}(-s+k)\times \left[\sum_{l=0}^{n-k-1}\binom{n}{l}z^{-(n-l)}(-b)^l\right]z^{s-k} +(-b+z^{-1})^n\binom{s}{n-1}b^{s-n+1}z^{s+1}$$
$$=1+\frac{(-1)^n}{b^{s+1}}\left[\binom{s}{n-1}-\binom{s+1}{n-1}\binom{n}{1} +\binom{s+2}{n-1}\binom{n}{2}+\cdots+(-1)^{n-1}\binom{s+n-1}{n-1}\binom{n}{n-1} -(-1)^{n-1}\binom{s+n}{n-1}\binom{n}{n-1}\right]z^{-s}$$
$$+\sum_{k=0}^{n-1}\alpha_b^{-n}(-(s+1)+k)\times \left[\sum_{l=0}^{n-k-1}\binom{n}{l}z^{-(n-l)}(-b)^l\right]z^{(s+1)-k}$$
$$=1+\sum_{k=0}^{n-1}\alpha_b^{-n}(-(s+1)+k)\times \left[\sum_{l=0}^{n-k-1}\binom{n}{l}z^{-(n-l)}(-b)^l\right]z^{(s+1)-k},$$

where the last equality arises from that the coefficient of z^{-s} is equal to 0 according to Lemma 5. Consequently, Eq. (26) also holds for $N = s+1$, and such an identity is valid for all $N \geq (n+1)$ by induction. □

Proof of Lemma 9:

Since $Z(C_b^n(N)) - 1 = \sum_{k=0}^{n-1} c$

$k) \left[\sum_{l=0}^{n-k-1} \binom{n}{l} z^{-(n-l)} (-b)^l \right] z^{N-k}$, we have

$$\|C_b^n(M) - \mathbf{1}\|_1$$
$$\leq \sum_{k=0}^{n-1} |\alpha_b^{-n}(-N+k)| \left[\left| \sum_{l=0}^{n-k-1} \binom{n}{l} (- \right.\right.$$
$$= \sum_{k=0}^{n-1} \left| \binom{N-k-1}{n-1} b^{N-k-n} \right| \left[\left| \sum_{l=0}^{n-k-1} \right.\right.$$
$$\leq \sum_{i=1}^{n} \left[\binom{N-i}{n-1} \sum_{k=0}^{n-i} \binom{n}{k} |b|^k \right] |b|^{N-n}$$

□

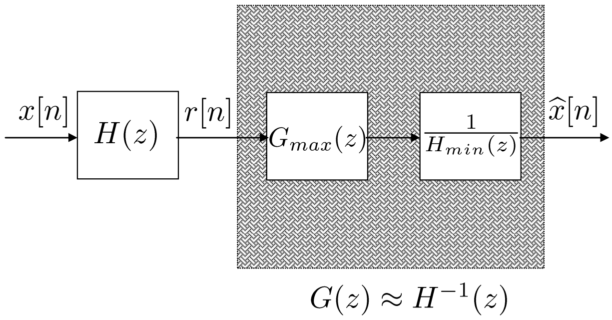

Fig. 1. Our proposed stabilized equalizer structure.

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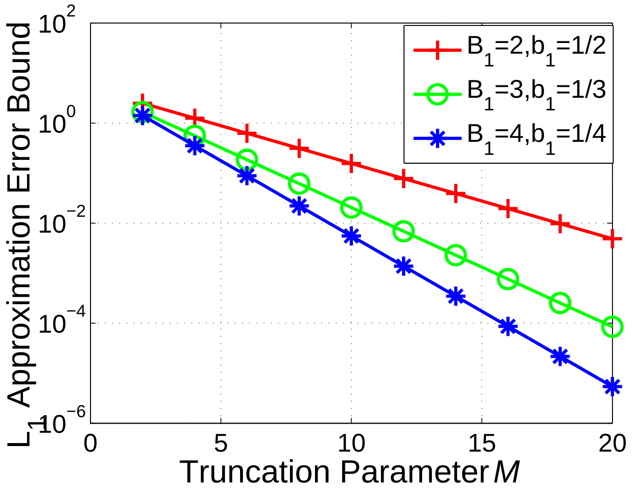

Fig. 2. L_1 approximation error bound versus truncation parameter M for different (B_1, b_1) with $N_1 = n_1 = 1$.

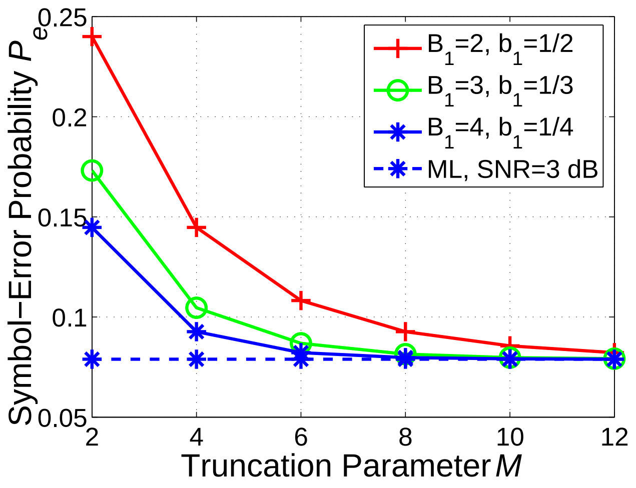

Fig. 4. Symbol-error probability P_e versus truncation parameter M for different (B_1, b_1) with $N_1 = n_1 = 1$.

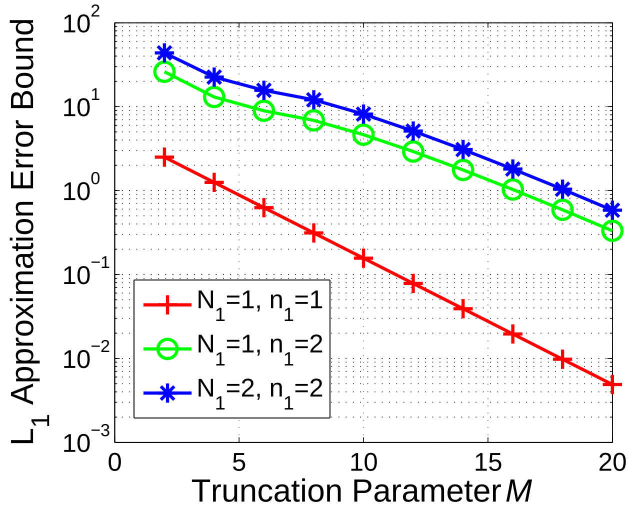

Fig. 3. L_1 approximation error bound versus truncation parameter M for different multiplicities (N_1, n_1).

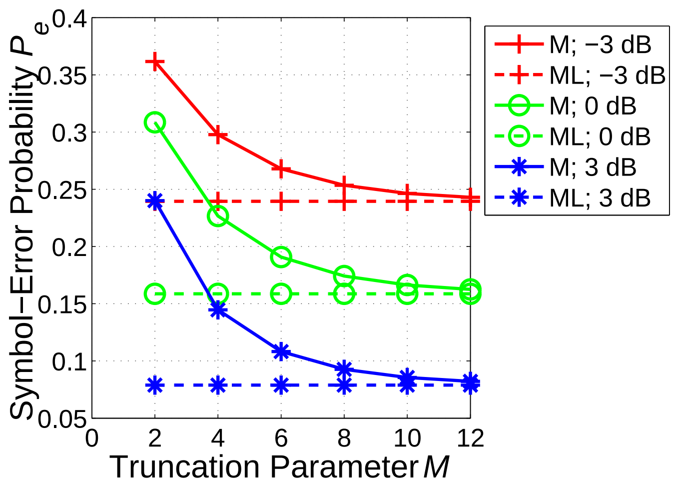

Fig. 5. Symbol-error probability P_e versus truncation parameter M for different SNRs.

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