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# Remarks on the Cauchy problem for multi-valued linear operators

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## 1. Introduction

Multi-valued linear operators appear, for example, as adjoints and closures of single-valued linear operators. Whereas these operators are always closed (see Section 2), degenerate Cauchy problems $Bu'(t) = Au(t),\ u(0) = x$ lead to possibly nonclosed multi-valued operators $\mathcal{A} = B^{-1}A$. In Section 2, we will extend the notion of *relatively closed* operators (see S.R. Caradus (1973) and B. Bäumer and F. Neubrander (1994a),(1994b)) to the multi-valued case. We will show that $\mathcal{A} = B^{-1}A$ is relatively closed even though $A$ and $B$ are not closed themselves and that sums and compositions of a single-valued and a multi-valued relatively closed operator are again relatively closed. In Section 3, mild and integrated solutions of the abstract Cauchy problem

$$u'(t) \in \mathcal{A}u(t),\ u(0) = x \qquad (0 \leq t \leq T_x)$$

will be characterized in terms of the approximate resolvent inclusion

$$x - a_k \in (kI - \mathcal{A})y(k), \qquad (k \in \mathbb{N},\ k \text{ sufficiently large}),$$

where $a_k$ is some sequence with exponential decay T (i.e., $\limsup_{k\to\infty} \frac{1}{k} \ln \|a_k\| \leq -T$). The results are used to give an alternative proof of A. Yagi's multi-valued version of the Hille-Yosida theorem and to extend R.S. Phillips' result for adjoint semigroups to nondensely defined Hille-Yosida operators. The results included are only a few of those possible. Their selection was done to show that most of the results on the abstract Cauchy problem

$$u'(t) = Au(t),\ u(0) = x \qquad (0 \leq t \leq T_x),$$

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where $A$ is a relatively closed single-valued linear operator (see B. Bäumer and F. Neubrander (1994a)(1994b)), extend naturally and without any major technical problems to the multi-valued linear case and, thus, to degenerate and adjoint Cauchy problems.

## 2. Multi-valued linear operators

Let $X, Y$ be Banach spaces. A multi-valued operator $\mathcal{A} \subset X \times Y$, where $\mathcal{A}x := \{y : (x, y) \in \mathcal{A}\}$, is called *linear* if the domain $D(\mathcal{A}) := \{x : (x, y) \in \mathcal{A} \text{ for some } y \in Y\}$ is a linear subspace and

$$\mu \mathcal{A}x \subset \mathcal{A}(\mu x) \text{ and } \mathcal{A}x + \mathcal{A}y \subset \mathcal{A}(x + y) \tag{2.1}$$

for all $x, y \in D(\mathcal{A})$ and $\mu \in \mathbb{C}$. Clearly $\mu \mathcal{A}x \subset \mathcal{A}(\mu x)$ implies that $\frac{1}{\mu}\mathcal{A}(\mu x) \subset \mathcal{A}x$, and thus $\mu \mathcal{A}x = \mathcal{A}(\mu x)$, $(\mu \in \mathbb{C}, x \in D(\mathcal{A}))$. Moreover, it follows from $\mathcal{A}x + \mathcal{A}y \subset \mathcal{A}(x+y)$, that $\mathcal{A}(x + y) - \mathcal{A}y = \mathcal{A}(x + y) + \mathcal{A}(-y) \subset \mathcal{A}x$ and hence $\mathcal{A}x + \mathcal{A}y = \mathcal{A}(x + y)$, $(x, y \in D(\mathcal{A}))$. This shows that, for $\mathcal{A} \subset X \times Y$, the statement (2.1) is equivalent to

$$\mu \mathcal{A}x = \mathcal{A}(\mu x) \text{ and } \mathcal{A}x + \mathcal{A}y = \mathcal{A}(x + y) \tag{2.2}$$

for all $x, y \in D(\mathcal{A})$ and $\mu \in \mathbb{C}$. The set $\mathrm{Im}(\mathcal{A}) := \{y : y \in \mathcal{A}x\,,\, x \in D(\mathcal{A})\}$ is called the image (or range) of the operator $\mathcal{A}$.

If a Banach space $Z$ is continuously embedded in a Banach space $X$, we will use the notation $Z \hookrightarrow X$. We say that $\mathcal{A} \subset X \times Y$ is *relatively closed* if there exist auxiliary Banach spaces $X_{\mathcal{A}}$ and $Y_{\mathcal{A}}$ such that

$$D(\mathcal{A}) \subset X_{\mathcal{A}} \hookrightarrow X\ ,\ \mathrm{Im}(\mathcal{A}) \subset Y_{\mathcal{A}} \hookrightarrow Y \tag{2.3}$$

and $\mathcal{A}$ is closed in $X_{\mathcal{A}} \times Y_{\mathcal{A}}$; i.e., $D(\mathcal{A}) \ni x_n \to x$ in $X_{\mathcal{A}}$ and $\mathcal{A}x_n \ni y_n \to y$ in $Y_{\mathcal{A}}$ implies that $x \in D(\mathcal{A})$ and $y \in \mathcal{A}x$. A relatively closed operator will also be called $X_{\mathcal{A}} \times Y_{\mathcal{A}}$-closed. If $A$ is a single-valued closed operator, then we denote by $[D(A)]$ the Banach space $D(A)$ endowed with the graph norm.

These definitions are motivated by the following examples (see also B. Bäumer and F. Neubrander (1994a), (1994b), A. Favini and A. Yagi (1993), and A. Yagi (1991)).

**Example 2.1 (Degenerate Cauchy Problem).** Let $A, B$ be single-valued linear operators with domains in a Banach space $X$ and ranges in a Banach space $Y$. Then

$$\mathcal{A} := B^{-1}A = \{(x, y) : x \in D(A), y \in D(B), \text{ and } Ax = By\}$$

is a possibly multi-valued operator in $X \times X$ with domain $D(\mathcal{A}) = \{x \in D(A) : Ax = By \text{ for some } y \in D(B)\}$. It is easy to see that $\mathcal{A}$ is linear. Since $D(\mathcal{A}) \ni x_n \to x$ and $\mathcal{A}x_n \ni y_n \to y$ is equivalent to $D(A) \ni x_n \to x$, $D(B) \ni y_n \to y$, and $Ax_n = By_n$, we obtain the following statements.

(2.1.1) If one of the operators $A, B$ is bounded and the other closed, then $\mathcal{A}$ is closed.

(2.1.2) If $A$ is closed and $B$ is $X_B \times Y$-closed, then $\mathcal{A}$ is $[D(A)] \times X_B$-closed.

(2.1.3) If $B$ is closed and $A$ is $X_A \times Y$-closed, then $\mathcal{A}$ is $X_A \times [D(B)]$-closed.

Consider the degenerate Cauchy problem

$$(DCP) \qquad Bu'(t) = Au(t),\ u(0) = x,\ 0 \le t \le T \le \infty.$$

A function $u \in C^1([0,T];X)$ with $u(t) \in D(A)$ and $u'(t) \in D(B)$ for $t \in [0,T]$ and which satisfies (DCP) is called a classical solution. Clearly $u$ is a classical solution of (DCP) if and only if $u$ is a solution of the abstract Cauchy problem

$$(ACP) \qquad u'(t) \in \mathcal{A}u(t),\ u(0) = x,\ 0 \le t \le T \le \infty,$$

where $\mathcal{A}$ is defined as above. Examples of concrete operators $\mathcal{A}$ to which (2.1.1)-(2.1.3) and (2.6.1) below can be applied can be found, for example, in A. Favini and A. Yagi (1993), N. Sauer (1982), and A. Yagi (1991). Assume that the operators $A, B$ are closed. In order to be able to integrate (DCP), we have to assume that $u \in C^1([0,T];X) \cap C([0,T];[D(A)])$ or, equivalently, that $u \in C^1([0,T];X)$ and $u' \in C([0,T];[D(B)])$. It follows from (2.1.2) and (2.1.3) above that $\mathcal{A}$ is $[D(A)] \times X$-closed as well as $X \times [D(B)]$-closed. Therefore, assuming that an operator $\mathcal{A}$ is $X_{\mathcal{A}} \times Y_{\mathcal{A}}$-closed, we will characterize in the next section the existence of those solutions of (ACP) which satisfy $u \in C([0,T];X_{\mathcal{A}})$ and $u' \in C([0,T];Y_{\mathcal{A}})$. The straightforward translation of the results for (ACP) in Section 3 into the corresponding statements for (DCP) is omitted. $\diamondsuit$

**Example 2.2 (Adjoint Operators).** Let $A$ be a single-valued linear operator with domain and range in a Banach space $X$. Define the adjoint of $A$ by

$$A^* := \{(x^*, y^*) : \langle x^*, Ax \rangle = \langle y^*, x \rangle \text{ for all } x \in D(A)\} \subset X^* \times X^*.$$

If $D(A)$ is dense in $X$, then $A^*$ is single-valued. In general, $A^*$ is always a closed and possibly multi-valued linear operator on $X^*$. To show the linearity, let $x^* \in D(A^*)$,

$\mu \in \mathbb{C}$, and $y^* \in \mu A^*(x^*)$. Then $\frac{1}{\mu}y^* \in A^*(x^*)$ implies that $\langle x^*, Ax\rangle = \langle \frac{1}{\mu}y^*, x\rangle$ or $\langle \mu x^*, Ax\rangle = \langle y^*, x\rangle$ for all $x \in D(A)$. Thus $y^* \in A^*(\mu x)$. Suppose $x_1^*, x_2^* \in D(A)$ and $y^* \in A^*(x_1^*) + A^*(x_2^*)$. Then $y^* = y_1^* + y_2^*$, where $\langle x_1^*, Ax\rangle = \langle y_1^*, x\rangle$ and $\langle x_2^*, Ax\rangle = \langle y_2^*, x\rangle$ for all $x \in D(A)$. Thus $\langle y^*, x\rangle = \langle y_1^* + y_2^*, x\rangle = \langle x_1^* + x_2^*, Ax\rangle$ for all $x \in D(A)$. Hence $y^* \in A^*(x_1^* + x_2^*)$. This shows that the adjoint $A^*$ is linear. To show the closedness of $A^*$, let $D(A^*) \ni x_n^* \to x^*$ and $A^*x^* \ni y_n^* \to y^*$. It follows from $\langle y_n^*, x\rangle = \langle x_n^*, Ax\rangle$, that $\langle y^*, x\rangle = \langle x^*, Ax\rangle$ for all $x \in D(A)$. Thus $x^* \in D(A^*)$ and $y^* \in A^*x^*$. $\diamondsuit$

**Example 2.3 (Closures of operators).** Let $\mathcal{A} \subset X \times Y$ be linear and let $\overline{\mathcal{A}}$ denote the closure of the graph of $\mathcal{A}$. We show that $\overline{\mathcal{A}}$ is linear. To see this let $x \in D(\overline{\mathcal{A}})$, $\mu \in \mathbb{C}$, and $y \in \mu\overline{\mathcal{A}}x$. Since $\frac{1}{\mu}y \in \overline{\mathcal{A}}x$, it follows that there exists $x_n \in D(\mathcal{A})$ such that $x_n \to x$ and $\mathcal{A}x_n \ni y_n \to \frac{1}{\mu}y$. Thus $\mu x_n \to \mu x$, and $\mathcal{A}(\mu x_n) \ni \mu y_n \to y$. This shows that $y \in \overline{\mathcal{A}}(\mu x)$. Let $y \in \overline{\mathcal{A}}(x_1) + \overline{\mathcal{A}}(x_2)$. Then there exists $y_1 \in \overline{\mathcal{A}}(x_1)$ and $y_2 \in \overline{\mathcal{A}}(x_2)$ such that $y = y_1 + y_2$, and there exist $v_n, u_n \in D(\mathcal{A})$ with $v_n \to x_1$, $\mathcal{A}v_n \ni z_n \to y_1$, $u_n \to x_2$, $\mathcal{A}u_n \ni w_n \to y_2$. Thus $v_n + u_n \to x_1 + x_2$ and $\mathcal{A}(v_n + u_n) \ni z_n + w_n \to y_1 + y_2 = y$. Therefore $y \in \overline{\mathcal{A}}(x_1 + x_2)$.

This shows that any linear operator has a closed linear extension. However, by considering the closure $\overline{\mathcal{A}}$ instead of $\mathcal{A}$, one might lose crucial information. To see this, consider $X = L^1[0,1]$, $A_k f := f^{(k)}(0) \cdot \mathbb{1}$, and $D(A_k) := C^{(k)}[0,1]$ $(k \in \mathbb{N}_0)$. Since there exist $f_i \in D(A_k)$ with $f_i \to 0$ in $L^1[0,1]$ and $A_k f_i = \mathbb{1}$ for all $i \in \mathbb{N}$, the operators $A_k$ do not have single-valued closures.

In fact, all the operators $A_k$ have the same closure $\mathcal{A} = \overline{A_k} = L^1[0,1] \times \mathbb{C} \cdot \mathbb{1}$ $(k \in \mathbb{N}_0)$. Because "closedness" is necessary for almost all operations in functional analysis and because $\mathcal{A} = \overline{A_k}$ does not contain any specific information on $A_k$, it is helpful to notice that the operators $A_k$ are $C^{(k)}[0,1] \times L^1[0,1]$-closed. $\diamondsuit$

**Example 2.4 (Sums).** Let $\mathcal{A}, \mathcal{B} \subset X \times Y$ be linear. Then the sum $\mathcal{S}x := \mathcal{A}x + \mathcal{B}x$ with $D(\mathcal{S}) := D(\mathcal{A}) \cap D(\mathcal{B})$ is linear. Let $A$ be single-valued and $X_A \times Y_A$-closed, and let $\mathcal{B}$ be $X_\mathcal{B} \times Y_\mathcal{B}$-closed where $Y_A \hookrightarrow Y_\mathcal{B}$. We show that $S := A + \mathcal{B}$ is $X_S \times Y_\mathcal{B}$-closed, where $X_S = D(A) \cap X_\mathcal{B}$, $\|x\|_{X_S} := \|x\|_{X_A} + \|x\|_{X_\mathcal{B}} + \|Ax\|_{Y_A}$. Clearly, $D(S) \subset X_S \hookrightarrow X$ and $\operatorname{Im}(S) \subset Y_A + Y_\mathcal{B} \subset Y_\mathcal{B} \hookrightarrow Y$. Suppose that $D(S) \ni x_n \to x$ in $X_S$ and $Sx_n \ni y_n \to y$ in $Y_\mathcal{B}$. Then there exists $z_n \in \mathcal{B}x_n$ such that $y_n = Ax_n + z_n \to y$ in $Y_\mathcal{B}$. Since $x_n \to x$ in $X_S$, it follows from the definition of the norm on $X_S$ and the $X_A \times Y_A$-closedness of $A$ that $x_n \to x$ in $X_A$ and $X_{\mathcal{B}}$, $x \in D(A)$, $Ax_n \to Ax$ in $Y_A$ and thus in $Y_{\mathcal{B}}$, since $Y_A \hookrightarrow Y_{\mathcal{B}}$. Since $x_n \to x$ in $X_{\mathcal{B}}$ and $\mathcal{B}x_n \ni z_n \to y - Ax$ in $Y_{\mathcal{B}}$, we conclude from the $X_{\mathcal{B}} \times Y_{\mathcal{B}}$-closedness of $\mathcal{B}$ that $x \in D(\mathcal{B})$ and $y - Ax \in \mathcal{B}x$. This shows that $x \in D(A) \cap D(\mathcal{B}) = D(S)$ and $y \in Ax + \mathcal{B}x = S(x)$. $\diamondsuit$

**Example 2.5 (Compositions).** Let $\mathcal{A} \subset X \times Y$ and $\mathcal{B} \subset Y \times Z$ be linear. It is easy to verify that the composition $\mathcal{C} = \mathcal{B}\mathcal{A} = \{(x, z) : x \in D(\mathcal{A}),\ z \in \mathcal{B}y$ for some $y \in Ax \cap D(\mathcal{B})\} \subset X \times Z$ is linear. Let $A$ be single-valued and $X_A \times Y_A$-closed and let $\mathcal{B}$ be $Y_{\mathcal{B}} \times Z_{\mathcal{B}}$-closed, where $Y_{\mathcal{B}} \hookrightarrow Y_A$. We show that $C$ is $X_C \times Z_{\mathcal{B}}$-closed, where $X_C = \{x \in D(A) : Ax \in Y_{\mathcal{B}},\ \|x\|_{X_C} := \|x\|_{X_A} + \|Ax\|_{Y_{\mathcal{B}}}\}$. It follows from the $X_A \times Y_A$-closedness of $A$ and $Y_{\mathcal{B}} \hookrightarrow Y_A$ that $X_C$ is a Banach space. Clearly, $D(C) \subset X_C \hookrightarrow X$. Let $D(C) \ni x_n \to x$ in $X_C$ and $Cx_n = \mathcal{B}Ax_n \ni z_n \to z$ in $Z_{\mathcal{B}}$. It follows from the $X_A \times Y_A$-closedness of $A$, $Y_{\mathcal{B}} \hookrightarrow Y_A$, and the definition of the norm in $X_C$ that $x \in D(A)$ and $Ax_n = y_n \to Ax = y$ in $Y_{\mathcal{B}}$. Since $\mathcal{B}$ is $Y_{\mathcal{B}} \times Z_{\mathcal{B}}$-closed, and $\mathcal{B}y_n \ni z_n \to z$ in $Z_{\mathcal{B}}$, we obtain that $Ax = y \in D(\mathcal{B})$ and $z \in \mathcal{B}Ay$. $\diamondsuit$

We remark that the results of the previous two examples can be extended to arbitrary finite and infinite sums and compositions as well as to limits of operators by using the results in B. Bäumer and F. Neubrander (1994b).

**Example 2.6.** With the result of the previous example, we can add one more closedness property of the operator $\mathcal{A} = B^{-1}A$ to (2.1.1)-(2.1.3). Let $A, B$ be linear operators with domains in a Banach space $X$ and ranges in a Banach space $Y$.

(2.6.1) If $A$ is $X_A \times Y_A$-closed and $B$ is $X_B \times Y_B$-closed, where $Y_B \hookrightarrow Y_A$, then $\mathcal{A}$ is $X_C \times X_B$-closed, where $X_C := \{x \in D(A) : Ax \in Y_B,\ \|x\|_{X_C} := \|x\|_{X_A} + \|Ax\|_{Y_B}$.

$\diamondsuit$

The following is a straightforward extension of a classical result of functional analysis due to E. Hille (see E. Hille and R.S.Phillips, 3.3.2 and 3.7.12).

**Proposition 2.7.** *Let $\mathcal{A} \subset X \times Y$ be linear and closed. If $u(\cdot) : [a, b] \to X$ is Bochner integrable, $u(t) \in D(\mathcal{A})$ for almost all $t \in [a, b]$, and $\mathcal{A}u(\cdot) \ni y(\cdot) : [a, b] \to Y$ is Bochner integrable, then*

$$\int_a^b y(s)\, ds \in D(\mathcal{A}) \text{ and } \int_a^b y(s)\, ds \in \mathcal{A} \int_a^b u(s)\, ds.$$

## 3. The abstract Cauchy problem

Let $X, Y$ be a Banach spaces and $\mathcal{A} \subset X \times X$ be linear and $X_{\mathcal{A}} \times Y_{\mathcal{A}}$-closed. Recall that $D(\mathcal{A}) \subset X_{\mathcal{A}} \hookrightarrow X$ and $\mathrm{Im}(\mathcal{A}) \subset Y_{\mathcal{A}} \hookrightarrow X$. Let $u \in C^1([0,T]; Y_{\mathcal{A}}) \cap C([0,T]; X_{\mathcal{A}})$, $u(0) = x$, $u(t) \in D(\mathcal{A})$, and $u'(t) \in \mathcal{A}u(t)$ for all $0 \leq t \leq T$. It follows from Proposition 2.7 that

$$(ACP_0) \qquad u(t) - x \in \mathcal{A} \int_0^t u(s)\, ds \quad (0 \leq t \leq T).$$

Integrating $(ACP_0)$ n-times yields

$$(ACP_n) \qquad v(t) - \frac{t^n}{n!} x \in \mathcal{A} \int_0^t v(s)\, ds \quad (0 \leq t \leq T),$$

where $v$ is the n-th normalized antiderivative

$$u^{[n]}(t) := \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} u(s)\, ds$$

of the solution $u$ of $(ACP_0)$. Let $v$ solve $(ACP_n)$ (i.e., $v$ is an integrated solution). If $v$ and $v^{[1]}$ are n-times differentiable in $Y_{\mathcal{A}}$ and $X_{\mathcal{A}}$ respectively, then $u = v^{[n]}$ solves $(ACP_0)$ (i.e., $u$ is a mild solution).

For $v : [0,T] \to X$, let $v_T : [0,\infty) \to X$ be defined as $v$ on $[0,T]$ and as 0 on $(T,\infty)$. For $v \in L^1_{loc}([0,\infty); X)$, we denote by $\mathrm{abs}_X(v)$ the infimum of all $a \in \mathbb{R}$ for which the Laplace transform

$$\hat{v}(\lambda) := X - \lim_{r \to \infty} \int_0^r e^{-\lambda t} v(t)\, dt$$

of $v$ exists for $\mathrm{Re}\lambda > a$. If $\omega \geq 0$, then $\mathrm{abs}_X(v) \leq \omega$ if and only if $\omega_X(v^{[1]}) \leq \omega$, where $\omega_X(v^{[1]}) := \inf\{\omega : \|v^{[1]}(t)\|_X \leq M_\omega e^{\omega t}$ for some $M_\omega > 0$ and all $t \geq 0\}$. For a proof, see, for example, G. Doetsch (1950), Satz 7.2.2. Notice that $\mathrm{abs}_X(v_T) = \omega_X(v_T) = -\infty$ if $T < \infty$ and that we substitute the interval $[0,\infty)$ for the interval $[0,T]$ if $T = \infty$.

We recall that a sequence $a_k \in X$ $(k \in \mathbb{N})$ is said to be of exponential decay $T$ if $\limsup_{k\to\infty} \frac{1}{k} \ln \|a_k\|_X \leq -T$. If $T = \infty$, then we set $a_k := 0$ $(k \in \mathbb{N})$.

The equivalence of the statements (i) and (iii) in the following lemma is the main observation in this section. It generalizes similar results in B. Bäumer and F. Neubrander (1994a), (1994b) in the single-valued case.

**Fundamental Lemma 3.1.** *Let $\mathcal{A} \subset X \times X$ be linear and $X_{\mathcal{A}} \times Y_{\mathcal{A}}$-closed, $x \in X$, $n \in \mathbb{N}_0$, $0 < T \leq \infty$, and $\omega \geq 0$. Let $v \in C([0,T]; Y_{\mathcal{A}})$ with $v^{[1]} \in C([0,T]; X_{\mathcal{A}})$, $abs_{Y_{\mathcal{A}}}(v_T) \leq \omega$, and $abs_{X_{\mathcal{A}}}(v_T^{[1]}) \leq \omega$. The following are equivalent:*

*(i)* $\int_0^t v(s)\, ds \in D(\mathcal{A})$ *and* $v(t) - \frac{t^n}{n!} x \in \mathcal{A} \int_0^t v(s)\, ds$ *for all* $t \in [0,T]$.

*(ii)* $\hat{v}_T(\lambda) \in D(\mathcal{A})$ *and* $x - e^{-\lambda T} \left[ \lambda^n v(T) + \sum_{i=0}^{n-1} \frac{(\lambda T)^i}{i!} x \right] \in (\lambda - \mathcal{A})(\lambda^n \hat{v}_T(\lambda))$ *if* $Re\lambda > \omega$.

*(iii)* $\hat{v}_T(k) \in D(\mathcal{A})$ *and there exists a sequence* $a_k \in Y_{\mathcal{A}}$ *of exponential decay* $T$ *such that* $x - a_k \in (kI - \mathcal{A})(k^n \hat{v}_T(k))$ *for all* $\omega < k \in \mathbb{N}$.

PROOF. (i)$\Rightarrow$ (ii). It follows from Proposition 2.7 that

$$\int_0^T e^{-\lambda t} v(t)\, dt - \int_0^T e^{-\lambda t} \frac{t^n}{n!} x\, dt \in \mathcal{A} \int_0^T e^{-\lambda t} v^{[1]}(t)\, dt.$$

Since $\int_0^T e^{-\lambda t} \frac{t^n}{n!} x = \frac{1}{\lambda^{n+1}} x - \frac{e^{-\lambda T}}{\lambda^{n+1}} \sum_{i=0}^n \frac{(\lambda T)^i}{i!} x$ and $\int_0^T e^{-\lambda t} v^{[1]}(t)\, dt = -\frac{1}{\lambda} e^{-\lambda T} v^{[1]}(T)$ $+\frac{1}{\lambda} \hat{v}_T(\lambda)$, we obtain that

$$\hat{v}_T(\lambda) - \frac{1}{\lambda^{n+1}} x + \frac{e^{-\lambda T}}{\lambda^{n+1}} \sum_{i=0}^n \frac{(\lambda T)^i}{i!} x \in \mathcal{A} \left( -\frac{1}{\lambda} e^{-\lambda T} v^{[1]}(T) + \frac{1}{\lambda} \hat{v}_T(\lambda) \right)$$

if Re$\lambda > \omega$. Since $v(T) - \frac{T^n}{n!} x \in \mathcal{A} v^{[1]}(T)$, we obtain that $\frac{1}{\lambda} e^{-\lambda T} v(T) - \frac{1}{\lambda} e^{-\lambda T} \frac{T^n}{n!} x \in \mathcal{A} \left( \frac{1}{\lambda} e^{-\lambda T} v^{[1]}(T) \right)$. Hence for Re$\lambda > \omega$

$$\hat{v}_T(\lambda) - \frac{1}{\lambda^{n+1}} x + \frac{1}{\lambda} e^{-\lambda T} v(T) + \frac{e^{-\lambda T}}{\lambda^{n+1}} \sum_{i=0}^n \frac{(\lambda T)^i}{i!} x \in \mathcal{A} \left( \frac{1}{\lambda} \hat{v}_T(\lambda) \right).$$

(ii)$\Rightarrow$(iii). Define $a_k := e^{-kT} \left[ k^n v(T) + \sum_{i=0}^{n-1} \frac{(kT)^i}{i!} x \right]$. Since $v(T) \in Y_{\mathcal{A}}$ and Im$\mathcal{A} \subset Y_{\mathcal{A}}$ it follows from $v(T) - \frac{T^n}{n!} x \in \mathcal{A} v^{[1]}(T)$ that $x \in Y_{\mathcal{A}}$. Thus $a_k \in Y_{\mathcal{A}}$. Obviously $a_k$ is of exponential decay T.

(iii)$\Rightarrow$(i). We will use the following *Phragmén-Doetsch inversion* of Laplace transform theory. For a proof, see B. Bäumer and F. Neubrander (1994a). If $r(\lambda) = \int_0^\infty e^{-\lambda t}\, df(t)$ for some $f \in \mathrm{Lip}_\omega([0,\infty); X) := \{f : [0,\infty] \to X : f(0) = 0 \text{ and } \|f\|_L < \infty\}$, where $\|f\|_L := \inf\{M : \|f(t) - f(s)\| \leq M \int_s^t e^{\omega r}\, dr \text{ for all } 0 \leq s \leq t\}$, then for all $k > \omega$

$$\|f(t) - \sum_{j=1}^\infty \frac{(-1)^{j+1}}{j!} e^{tjk} r(kj)\| \leq \frac{2}{k} \|f\|_L. \tag{3.1}$$

Let $\omega' > \omega$. Then $v_T^{[2]} \in \mathrm{Lip}_{\omega'}([0,\infty); Y_{\mathcal{A}}),\ v_T^{[3]} \in \mathrm{Lip}_{\omega'}([0,\infty); X_{\mathcal{A}}),$ and

$$y(\lambda) := \lambda^n \hat{v}_T(\lambda) = \lambda^{n+1} \int_0^\infty e^{-\lambda t}\, dv_T^{[2]}(t) = \lambda^{n+2} \int_0^\infty e^{-\lambda t}\, dv_T^{[3]}(t).$$

Since $Y_{\mathcal{A}} - \lim_{k\to\infty} \sum_{j=1}^\infty \frac{(-1)^{j+1}}{j!} e^{tjk} \frac{1}{(kj)^{n+2}} a_{kj} = 0$ for $0 \le t < T$ and $z(k) := ky(k) - x + a_k \in \mathcal{A}y(k)$ for sufficiently large $k \in \mathbb{N}$, it follows from the $X_{\mathcal{A}} \times Y_{\mathcal{A}}$-closedness of $\mathcal{A}$ that $v_T^{[3]}(t) \in D(\mathcal{A})$ and

$$\begin{aligned} v_T^{[2]}(t) - \frac{t^{n+2}}{(n+2)!}x &= Y_{\mathcal{A}} - \lim_{k\to\infty} \sum_{j=1}^\infty \frac{(-1)^{j+1}}{j!} e^{tjk} \left( \frac{y(kj)}{(kj)^{n+1}} - \frac{x}{(kj)^{n+2}} + \frac{a_{kj}}{(kj)^{n+2}} \right) \\ &= Y_{\mathcal{A}} - \lim_{k\to\infty} \sum_{j=1}^\infty \frac{(-1)^{j+1}}{j!} e^{tjk} \frac{1}{(kj)^{n+2}} z(kj) \\ &\in \mathcal{A} \left( X_{\mathcal{A}} - \lim_{k\to\infty} \sum_{j=1}^\infty \frac{(-1)^{j+1}}{j!} e^{tjk} \frac{y(kj)}{(kj)^{n+2}} \right) = \mathcal{A} v_T^{[3]}(t) \end{aligned}$$

for $0 \le t < T$ and thus for $t \in [0,T]$. Now (i) follows from the $X_{\mathcal{A}} \times Y_{\mathcal{A}}$-closedness of $\mathcal{A}$ and the differentiability of $v_T^{[2]}$ and $v_T^{[3]}$. ♢

As the Fundamental Lemma 3.1 shows, it is necessary for the existence of a solution of $(ACP_n)$ for some initial value $x \in X$ that $x$ is in the *approximate range of* $\mathcal{A}$; i.e., there exist $\omega \ge 0$, an error function $\lambda \to a(\lambda)$ of exponential decay $T$ and $y(\lambda) \in D(\mathcal{A})$ such that $x - a(\lambda) \in (\lambda - \mathcal{A})y(\lambda)$ for all $\lambda \ge \omega$. As in the single-valued case, we obtain the following Hille-Yosida type characterization from Laplace transform theory (see also B. Bäumer and F. Neubrander (1994a), (1994b)). Let $\Omega \subset \mathbb{C}$ (or $\mathbb{R}$). We say that $q : \Omega \to X$ is polynomialy bounded in $X$ if there exists a polynomial $p$ such that $\|q(\lambda)\|_X \le p(|\lambda|)$ for $\lambda \in \Omega$.

**Theorem 3.2 (Existence and Uniqueness).** *Let $\mathcal{A} \subset X \times X$ be linear and $X_{\mathcal{A}} \times Y_{\mathcal{A}}$-closed, $x \in X$, and $0 < T \le \infty$. The following are equivalent.*

*(i) There exists $n \in \mathbb{N}_0$ and $v \in C([0,T]; Y_{\mathcal{A}})$ with $v^{[1]} \in C([0,T]; X_{\mathcal{A}})$, $\mathrm{abs}_{Y_{\mathcal{A}}}(v_T) < \infty$, $\mathrm{abs}_{X_{\mathcal{A}}}(v_T^{[1]}) < \infty$, $\int_0^t v(s)\,ds \in D(\mathcal{A})$, and $v(t) - \frac{t^n}{n!}x \in \mathcal{A} \int_0^t v(s)\,ds$.*

*(ii) There exists a sequence $a_k \in Y_A$ of exponential decay $T$ and, for some $\omega \in \mathbb{R}$, there exists a function $y : \{Re\lambda > \omega\} \to D(\mathcal{A}) \cap Y_{\mathcal{A}}$ which is analytic and polynomially bounded in $X_{\mathcal{A}}$ and $Y_{\mathcal{A}}$ such that $x - a_k \in (k - \mathcal{A})y(k)$ for sufficiently large $k \in \mathbb{N}$.*

Let $\omega \geq 0$ and $r : (\omega, \infty) \to X$. We say that $r$ has a Laplace representation in $\mathrm{Lip}_\omega(X)$ if there exists $v \in \mathrm{Lip}_\omega([0,\infty); X)$ such that $r(\lambda) = \lambda \int_0^\infty e^{-\lambda t} v(t)\, dt = \int_0^\infty e^{-\lambda t}\, dv(t)$ for all $\lambda > \omega$. For a proof of the following facts from Laplace transform theory, see B. Bäumer and F. Neubrander (1994a).

**Proposition 3.3 (Laplace Representation).** *Let $\omega \geq 0$ and $r : (\omega, \infty) \to X$. The following are equivalent:*

*(i) $r$ has a Laplace representation in $Lip_\omega(X)$.*
*(ii) $r \in C^\infty((\omega,\infty); X)$ and $\sup_{\lambda>\omega,\, k\in\mathbb{N}_0} \|(\lambda-\omega)^{k+1} \frac{1}{k!} r^{(k)}(\lambda)\| < \infty$.*
*(iii) $r : \{Re\lambda > \omega\} \to X$ is analytic, $r(\lambda) \to 0$ as $\lambda \to \infty$, $\sup\{\|r(\lambda)\| : Re\lambda > \gamma\} < \infty$ for all $\gamma > \omega$, and there exists $k_0 \in \mathbb{N}_0$ such that for all $k \geq k_0$*

$$\sup_{\gamma>\omega,\, s>0} \left\| \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{r(\gamma+it)}{(1-ist)^{k+2}}\, dt \right\| < \infty.$$

*(iv) There exists $M > 0$ such that $\left\| \sum_{j=1}^n \lambda_j r(j+\omega) \right\| \leq M \int_0^\infty \left| \sum_{j=1}^n \lambda_j e^{-jt} \right| dt$ for all $n \in \mathbb{N}$ and $\lambda_j \in \mathbb{C}$.*

Combining Proposition 3.3 with the Fundamental Lemma 3.1, we obtain the following local Hille-Yosida theorem.

**Theorem 3.4 (Local Hille-Yosida Theorem).** *Let $\mathcal{A} \subset X \times X$ be linear and $X_\mathcal{A} \times Y_\mathcal{A}$-closed, $x \in X$, $n \in \mathbb{N}$ and $\omega \geq 0$. The following are equivalent:*

*(i) There exists $v \in Lip_\omega([0,\infty); Y_\mathcal{A})$ with $v^{[1]} \in Lip_\omega([0,\infty); X_\mathcal{A})$, $\int_0^t v(s)\, ds \in D(\mathcal{A})$, and $v(t) - \frac{t^n}{n!} x \in \mathcal{A} \int_0^t v(s)\, ds$ for all $t \geq 0$.*
*(ii) There exists $y : (\omega, \infty) \to D(\mathcal{A}) \cap Y_\mathcal{A}$ such that*
*(a) $x \in (kI - \mathcal{A}) y(k)$ for $\omega < k \in \mathbb{N}$,*
*(b) $\lambda \to \frac{1}{\lambda^{n-1}} y(\lambda)$ has a Laplace representation in $Lip_\omega(Y_\mathcal{A})$,*
*(c) $\lambda \to \frac{1}{\lambda^n} y(\lambda)$ has a Laplace representation in $Lip_\omega(X_\mathcal{A})$.*

*Moreover, if $Y_\mathcal{A} \hookrightarrow X_\mathcal{A}$, then condition (c) can be dropped.*

Clearly, Theorem 3.4 can be used to prove global Hille-Yosida type theorems characterizing those Cauchy problems $(ACP_n)$ which have solutions for all $x \in X_0 \subset X$ and not just for a single $x \in X$. We demonstrate this with a Hille-Yosida theorem for multi-valued linear operators due to A. Yagi (1991); see Theorem 3.6 below.

Let $\mathcal{A} \subset X \times X$ be linear. The set of all $\lambda \in \mathbb{C}$ for which $\operatorname{Im}(\lambda - \mathcal{A}) = D((\lambda - \mathcal{A})^{-1}) = X$ and $R(\lambda, \mathcal{A}) := (\lambda - \mathcal{A})^{-1}$ is a single-valued, bounded operator on $X$ is called the *resolvent set* $\rho(\mathcal{A})$ of $\mathcal{A}$. It easy to see that any $\mathcal{A} \subset X \times X$ with $\rho(\mathcal{A}) \neq \emptyset$ is closed. Clearly,

$$\text{if } \lambda \in \rho(\mathcal{A}) \text{ and } 0 \in (\lambda - \mathcal{A})x, \text{ then } x = 0. \tag{3.2}$$

Let $\lambda \in \rho(\mathcal{A})$ and $y \in \mathcal{A}x = \lambda x - (\lambda - \mathcal{A})x$. Then $R(\lambda, \mathcal{A})y = \lambda R(\lambda, \mathcal{A})x - x$. Thus

$$R(\lambda, \mathcal{A})\mathcal{A}x = \lambda R(\lambda, \mathcal{A})x - x \qquad (x \in D(\mathcal{A}),\ \lambda \in \rho(\mathcal{A})). \tag{3.3}$$

In particular, $R(\lambda, \mathcal{A}) = 0$ on the linear subspace $\mathcal{A}0$. Since $x = R(\lambda_0, \mathcal{A})(\lambda_0 - \mathcal{A})x$ $(x \in D(\mathcal{A}),\ \lambda_0 \in \rho(\mathcal{A}))$, it follows that $(\lambda - \mathcal{A})x = -(\lambda_0 - \lambda)x + (\lambda_0 - \mathcal{A})x = -(\lambda_0 - \lambda)R(\lambda_0, \mathcal{A})(\lambda_0 - \mathcal{A})x + (\lambda_0 - \mathcal{A})x = (I - (\lambda_0 - \lambda)R(\lambda_0, \mathcal{A}))(\lambda_0 - \mathcal{A})x$. Thus, $\rho(\mathcal{A})$ is open and, if $\lambda_0 \in \rho(\mathcal{A})$ and $|\lambda - \lambda_0| \|R(\lambda_0, \mathcal{A})\| < 1$, then

$$R(\lambda, \mathcal{A}) = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n R(\lambda_0, \mathcal{A})^{n+1}. \tag{3.4}$$

It follows that the resolvent is analytic on $\rho(\mathcal{A})$ and

$$(-1)^n \frac{1}{n!} R^{(n)}(\lambda_0, \mathcal{A}) = R(\lambda_0, \mathcal{A})^{n+1} \qquad (n \in \mathbb{N},\ \lambda_0 \in \rho(\mathcal{A})). \tag{3.5}$$

Let $\lambda, \mu \in \rho(\mathcal{A})$. Since $x \in (\mu - \mathcal{A})R(\mu, \mathcal{A})x$ $(x \in X)$ and $x = R(\lambda, \mathcal{A})(\lambda - \mathcal{A})x$ $(x \in D(\mathcal{A}))$, it follows that $R(\lambda, \mathcal{A})x - R(\mu, \mathcal{A})x \subset R(\lambda, \mathcal{A})(\mu - \mathcal{A})R(\mu, \mathcal{A})x - R(\lambda, \mathcal{A})(\lambda - \mathcal{A})R(\mu, \mathcal{A})x$. By (3.3), $R(\lambda, \mathcal{A})\mathcal{A}R(\mu, \mathcal{A})$ is single-valued and thus $R(\lambda, \mathcal{A})$ satisfies the resolvent equation

$$R(\lambda, \mathcal{A}) - R(\mu, \mathcal{A}) = R(\lambda, \mathcal{A})(\mu - \lambda)R(\mu, \mathcal{A}) \qquad (\lambda, \mu \in \rho(\mathcal{A})). \tag{3.6}$$

Before we discuss the Hille-Yosida theorem, we will extend Lyubich's Uniqueness theorem (see, for example, A. Pazy, Theorem 4.1.2) to the multi-valued case.

**Theorem 3.5 (Lyubich's Uniqueness Theorem).** *Let $\mathcal{A} \subset X \times X$ be linear. If $R(k, \mathcal{A})$ exists for all $k_0 \leq k \in \mathbb{N}$ and $\limsup_{k\to\infty} \frac{1}{k} \ln \|R(k, \mathcal{A})\| \leq 0$, then $(ACP_0)$ has at most one solution for every $x \in X$.*

PROOF. Let $v \in C([0,T];X)$, $\int_0^t v(s)\,ds \in D(\mathcal{A})$, and $v(t) \in \mathcal{A}\int_0^t v(s)\,ds$ for all $0 \leq t \leq T < \infty$. It follows from the Fundamental Lemma 3.1 that there exists a sequence $a_k$ with $\limsup_{k\to\infty} \frac{1}{k}\ln\|a_k\| \leq -T$ such that $-a_k \in (kI - \mathcal{A})\hat{v}_T(k)$ for all sufficiently large $k \in \mathbb{N}$. Thus $\limsup_{k\to\infty}\frac{1}{k}\ln\|\hat{v}_T(k)\| = \limsup_{k\to\infty}\frac{1}{k}\ln\|R(k,\mathcal{A})a_k\| \leq -T$. Let $\epsilon > 0$. Then there exists $k_\epsilon \geq k_0$ such that

$$\left\|\int_0^T e^{-kt}v(t)\,dt\right\| = \|\hat{v}_T(k)\| \leq e^{-(T-\epsilon)k}$$

and hence $\left\|\int_{-\epsilon}^{T-\epsilon} e^{ks}v(T-\epsilon-s)\,ds\right\| \leq 1$ for all $k \geq k_\epsilon$. Therefore there exists $M > 0$ such that $\left\|\int_0^{T-\epsilon} e^{ks}v(T-\epsilon-s)\,ds\right\| \leq M$ for all $k \geq k_\epsilon$. Now we obtain from Lemma 4.1.1 in A. Pazy (1983) that $v(t) = 0$ for all $t \in [0, T-\epsilon]$. $\diamondsuit$

**Theorem 3.6 (Global Hille-Yosida Theorem).** *Let $\mathcal{A} \subset X \times X$ be linear. Assume that there exist $M, \omega \in \mathbb{R}$ such that $(\omega,\infty) \subset \rho(\mathcal{A})$ and $\|R(\lambda,\mathcal{A})^n\| \leq \frac{M}{(\lambda-\omega)^n}$ for all $\lambda > \omega$ and $n \in \mathbb{N}$. Then*

*(a) The restriction $A$ of $\mathcal{A}$ to $\overline{D(\mathcal{A})} \times \overline{D(\mathcal{A})}$ is single-valued, $D(A) = \{x \in D(\mathcal{A}) : \mathcal{A}x \cap \overline{D(\mathcal{A})} \neq \emptyset\}$, and $Ax = y = \mathcal{A}x \cap \overline{D(\mathcal{A})}$,*

*(b) $A$ generates a strongly continuous semigroup on $\overline{D(\mathcal{A})}$,*

*(c) $(ACP_1)$ has a unique solution for all $x \in X$.*

PROOF. By (3.5) and the statements $(ii)$ in Proposition 3.3 and Theorem 3.4, there exists $S : [0,\infty) \to \mathcal{L}(X)$ with $S(0) = 0$, $\|S(t) - S(s)\| \leq M\int_s^t e^{\omega r}\,dr$ $(0 \leq s \leq t)$, and $R(\lambda,\mathcal{A}) = \lambda\int_0^\infty e^{-\lambda t}S(t)\,dt$ $(\lambda > \max\{\omega, 0\})$. Thus, by the Fundamental Lemma 3.1,

$$S(t)x - tx \in \mathcal{A}\int_0^t S(s)x\,ds \qquad (x \in X,\ t \geq 0). \tag{3.7}$$

This and Theorem 3.5 prove (c). Since $R(\lambda,\mathcal{A})y = R(\lambda,\mathcal{A})z$ for $y, z \in \mathcal{A}x$ (see (3.3)), it follows from the uniqueness of the Laplace transform that $S(t)y = S(t)z$. Thus

$$S(t)\mathcal{A} \text{ is single-valued.} \tag{3.8}$$

In particular, $S(t)$ vanishes on $\mathcal{A}0$. Let $x \in D(\mathcal{A})$ and $\lambda > \max\{\omega, 0\}$. Then by (3.3),

$$\begin{aligned}\lambda^2 \int_0^\infty e^{-\lambda t} tx\, dt &= \lambda R(\lambda, \mathcal{A})x - R(\lambda, \mathcal{A})\mathcal{A}x \\ &= \lambda^2 \int_0^\infty e^{-\lambda t} S(t)x\, dt - \lambda \int_0^\infty e^{-\lambda t} S(t)\mathcal{A}x\, dt \\ &= \lambda^2 \int_0^\infty e^{-\lambda t} \left[ S(t)x - \int_0^t S(s)\mathcal{A}x\, ds \right] dt.\end{aligned}$$

By the uniqueness of the Laplace transform,

$$S(t)x - tx = \int_0^t S(s)\mathcal{A}x\, ds \qquad (t \geq 0, x \in D(\mathcal{A})). \tag{3.9}$$

It follows that $S(\cdot)x$ is continuously differentiable on $[0, \infty)$ for all $x \in D(\mathcal{A})$. Since the differential quotients $\frac{1}{h}(S(t+h) - S(t))$ are uniformly bounded for $0 < h \leq 1$, we obtain that $S(t)x$ is differentiable for all $x \in \overline{D(\mathcal{A})}$. Define $T(t)x := S'(t)x$ for $x \in \overline{D(\mathcal{A})}$. Then $\|T(t)\| \leq Me^{\omega t}$. Since $t \to T(t)x$ is continuous for all $x \in D(\mathcal{A})$ and $\|T(t)\| \leq Me^{\omega t}$ it follows that $T(t)$ is strongly continuous on $\overline{D(\mathcal{A})}$. We conclude from $T(t)x - x = S(t)\mathcal{A}x$ ($x \in D(\mathcal{A})$, $t \geq 0$) and $S(0) = 0$ that $T(0) = I$ on $\overline{D(\mathcal{A})}$. Since $R(\lambda, \mathcal{A})x = \int_0^\infty e^{-\lambda t}\, dS(t)x$ for all $\lambda > \omega$ and $x \in X$, we obtain

$$R(\lambda, \mathcal{A})x = \int_0^\infty e^{-\lambda t} T(t)x\, dt \qquad (x \in \overline{D(\mathcal{A})}, \lambda > \omega). \tag{3.10}$$

We remark that the resolvent of a multi-linear operator is in general not one-to-one. However, by (3.10), if $R(\lambda, \mathcal{A})x = 0$ for some $x \in \overline{D(\mathcal{A})}$, then $T(t)x = 0$ for all $t \geq 0$. In particular, $x = T(0)x = 0$. This shows that $R(\lambda, \mathcal{A})$ is one-to-one on $\overline{D(\mathcal{A})}$. Therefore, employing a standard result for pseudo-resolvents, we conclude that there exists an operator $A$ such that $R(\lambda, A)x = R(\lambda, \mathcal{A})x = \int_0^\infty e^{-\lambda t} T(t)x\, dt$ for all $x \in \overline{D(\mathcal{A})}$ and $\lambda > \omega$. It follows from (3.1) that

$$S(t)x \in \overline{D(\mathcal{A})} \text{ for all } x \in X. \tag{3.11}$$

Thus $T(t) \in \mathcal{L}(\overline{D(\mathcal{A})})$. This implies that $T(t)$ is a $C_0$-semigroup on $\overline{D(\mathcal{A})}$ with generator $A$ (see, for example W. Arendt (1987)). By (3.7), $\frac{1}{t}(T(t)x - x) \in \mathcal{A}\left(\frac{1}{t}\int_0^t T(s)x\, ds\right)$ for all $x \in \overline{D(\mathcal{A})}$ and $t \geq 0$. Thus $D(A) \subset D(\mathcal{A})$ and $Ax \in \mathcal{A}x$. This shows that $D(A) \subset \{x \in D(\mathcal{A}) : \mathcal{A}x \cap \overline{D(\mathcal{A})} \neq \emptyset\}$. Suppose that $x \in D(\mathcal{A})$ and $y \in \mathcal{A}x \cap \overline{D(\mathcal{A})}$. Then $S(s)\mathcal{A}x = S(s)y$ for all $s \geq 0$. It follows from $T(t)x - x = S(t)y = \int_0^t T(s)y ds$ that $x \in D(A)$ and $Ax = y$. Thus $D(A) = \{x \in D(\mathcal{A}) : \mathcal{A}x \cap \overline{D(\mathcal{A})} \neq \emptyset\}$ and $Ax = y = \mathcal{A}x \cap \overline{D(\mathcal{A})}$. $\diamondsuit$

**Example 3.7 (Adjoint semigroups).** Let $A$ be a Hille-Yosida operator on a Banach space $X$; i.e., $A$ is single-valued, $(\omega, \infty) \subset \rho(A)$ and $\|R(\lambda, A)^n\| \leq \frac{M}{(\lambda-\omega)^n}$ $(\lambda > \omega,\, n \in \mathbb{N})$. Since $D(A)$ might not be dense, the adjoint $A^*$ might be multi-valued. It is easy to see that $(\omega, \infty) \subset \rho(A^*)$, $R(\lambda, A^*) = R(\lambda, A)^*$ for $\lambda > \omega$ and thus $\|R(\lambda, A^*)^n\| \leq \frac{M}{(\lambda-\omega)^n}$ $(\lambda > \omega,\, n \in \mathbb{N})$. Thus Theorem 3.6 can be applied for $\mathcal{A} := A^*$ and extends the well-known result of R.S. Phillips for adjoint semigroups to Hille-Yosida operators with non-dense domains (see A. Pazy, Theorem 1.10.4).

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