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DEGENERATE ABSTRACT CAUCHY PROBLEMS

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Abstract Laplace-Stieltjes transform methods are applied to obtain existence and uniqueness results for exponentially bounded solutions of degenerate abstract Cauchy problems $\frac{d}{dt}Bu(t) = Au(t)$, $Bu(0) = x$ or $Bu'(t) = Au(t)$, $u(0) = z$, where A, B are closed operators on a Banach space.

1. INTEGRAL SOLUTIONS

In this paper we reconsider the transform approach taken by A. Favini (1979) to study the degenerate abstract Cauchy problem or *Sobolev equation*

$$\frac{d}{dt}Bu(t) = Au(t)\ ;\ Bu(0) = x. \tag{SE}$$

Throughout the paper, A and B are closed linear operators whose ranges and domains $D(A), D(B)$ are contained in a Banach space E. The equation (SE) has been studied extensively by various authors. See, for example, R.W. Carroll and R.E. Showalter (1976), N. Sauer and J.E. Singleton (1989), and H. Fattorini (1983), where further references can be found. For $B = I$ the results of this section are contained in Neubrander (1992).

Besides closedness no other assumptions are made on the operators A, B in Section 1. In particular, it is not required that the operator pencil $\lambda B - A$ with domain $D(A) \cap D(B)$ is closable for any $\lambda \in \mathbb{C}$. As a consequence, only mild or integral solutions of (SE) are obtained. Among others, the results of the first section can be used to study multiplicative perturbations. Let A, M be closed with M being injective. In general, if A and M^{-1} are unbounded, then the multiplicative perturbation MA of the operator A with domain $D(MA) = \{x \in D(A) : Ax \in D(M)\}$ is not closed (and might not even be closable). Hence, none of the existing semigroup theories can be used to study the Cauchy problem

$u'(t) = MAu(t); u(0) = x; u(t) \in D(MA)$. However, if $B := M^{-1}$, then the Cauchy problem for MA can be transformed into a degenerate one.

In Section 2 it is assumed that $\lambda B - A$ is closed and has a bounded inverse $R(\lambda) := (\lambda B - A)^{-1}$ which satisfies $\|R(\lambda)\| + \|BR(\lambda)\| \le p(|\lambda|)$ for some polynomial p and all $\lambda \in \mathbb{C}$ with $Re\lambda > \omega$. It will be shown that there exists an $n \in \mathbb{N}_0$ such that

$$Bu'(t) = Au(t)\ ;\ u(0) = (R(\lambda_0)B)^n\, x \qquad (ACP_B)$$

has a classical solution for all $x \in D(A) \cap D(B) \cap D\left((R(\lambda_0)B)^n\right)$.

In order to formulate existence and uniqueness theorems for exponentially bounded solutions of (SE) we consider the following spaces of vector-valued functions.

For $\omega \ge 0$ let $Lip_\omega(E)$ be the space of all functions $v : [0, \infty) \to E$ with $v(0) = 0$ and $\|v(t+h) - v(t)\| \le hMe^{w(t+h)}$ for all $t, h \ge 0$ and some contant M. Then $Lip_\omega(E)$ is a Banach space with norm

$$\|v\|_{Lip(\omega)} \ := \ \inf\{M : \|v(t+h) - v(t)\| \le hMe^{\omega(t+h)} \ \text{ for all } \ t, h \ge 0\}.$$

For $\omega \ge 0$ let $C^\infty_\omega(E)$ be Widder's Banach space of all functions $r \in C^\infty((\omega, \infty), E)$ with norm

$$\|r\|_\omega \ := \ \sup_{k \in \mathbb{N}_0} \sup_{\lambda > \omega} \|\frac{1}{k!}(\lambda - \omega)^{k+1} r^{(k)}(\lambda)\| \ < \infty.$$

For $v \in Lip_\omega(E)$ the Laplace-Stieltjes transform $\mathcal{L}_S v := r$, where

$$r(\lambda) \ := \ \int_0^\infty e^{-\lambda t}\, dv(t) \text{ for all } \lambda > \omega,$$

is well defined. The function r is in $C^\infty_\omega(E)$ and the Laplace-Stieltjes transform $\mathcal{L}_S$ is an isometric isomorphism between the spaces $Lip_\omega(E)$ and $C^\infty_\omega(E)$. This is the *Widder-Arendt Representation Theorem* of Laplace-Stieltjes transform theory (see Arendt (1987), Hennig and Neubrander (1990) or Neubrander (1992)). For $r \in C^\infty_\omega(E)$ the inverse Laplace-Stieltjes transform is given by

$$(\mathcal{L}_S^{-1} r)(t) \ = \ v(t) \ = \ \lim_{n \to \infty} \frac{1}{2\pi i} \int_{\omega+1-in}^{\omega+1+in} e^{\lambda t} \frac{r(\lambda)}{\lambda}\, d\lambda$$

for all $t > 0$ (see, for example, Neubrander (1992)).

For some $\omega \geq 0$ the space $L^\infty_\omega(E)$ consists of all Bochner integrable functions $u : (0,\infty) \to E$ for which

$$\|u\|_{\infty,\omega} \;:=\; \sup_{t>0} \|e^{-\omega t} u(t)\| \;<\; \infty.$$

If the space E has the Radon-Nikodym property (like any reflexive Banach space), then the antiderivative operator $Iu \;:\; t \to \int_0^t u(s)\,ds$ is an isometric isomorphism between $L^\infty_\omega(E)$ and $Lip_\omega(E)$ and the Laplace transform $\mathcal{L}u := r$, where

$$r(\lambda) \;:=\; \int_0^\infty e^{-\lambda t} u(t)\,dt \text{ for all } \lambda > \omega,$$

is an isometric isomorphism between $L^\infty_\omega(E)$ and $C^\infty_\omega(E)$.

For a closed operator B on E the Banach space $[D(B)]$ consists of $D(B)$ with the graph norm $\|x\|^B := \|x\| + \|Bx\|$. Note that $u \in L^\infty_\omega([D(B)])$, $Lip_\omega([D(B)])$ or $C^\infty_\omega([D(B)])$ if and only if u and Bu are in $L^\infty_\omega(E)$, $Lip_\omega(E)$, or $C^\infty_\omega(E)$. The norms in the spaces $L^\infty_\omega([D(B)]), Lip_\omega([D(B)])$, or $C^\infty_\omega([D(B)])$ will be denoted by $\|u\|^B_{\infty,\omega}, \|v\|^B_{Lip(\omega)}$, and $\|r\|^B_\omega$.

A function u is a *classical solution* of (SE) if $u \in C([0,\infty),[D(A)]) \cap C([0,\infty),[D(B)])$, $Bu \in C^1([0,\infty),E)$, and (SE) holds for all $t \geq 0$.

Let u be a classical solution of (SE). Then

$$Bu(t) - x = A\int_0^t u(s)\,ds \tag{SE_0}$$

for all $t \geq 0$. A function $u \in L^\infty_\omega([D(B)])$ is called a mild exponential solution or just *mild solution* of (SE) if it satisfies (SE_0) for almost all $t \geq 0$.

Let u be a mild solution of (SE). Define $v(t) := \int_0^t u(s)\,ds$. Then $v \in Lip_\omega([D(B)])$ and $Bu(t) - x = Av(t)$. Integrating both sides from 0 to t yields

$$Bv(t) - tx \;=\; A\int_0^t v(s)\,ds \tag{SE_1}$$

for all $t \geq 0$. Continuing this process yields the n-th integrated Sobolev equation

$$Bv(t) - \frac{t^n}{n!}x \;=\; A\int_0^t v(s)\,ds. \tag{SE_n}$$

A function $v \in L^\infty_\omega([D(B)])$ is called a *mild n-th integral solution* of (SE) or a *mild solution* of (SE_n) if it satisfies (SE_n) for almost all $t \geq 0$.

Let $n \geq 1$. A function $v \in Lip_\omega([D(B)])$ is called a *strong n-th integral solution* of (SE) or a *strong solution* of (SE_n) if it satisfies (SE_n) for all $t \geq 0$.

If v is a mild solution of (SE_n), then $w\ :\ t \to \int_0^t v(s)ds$ is a strong solution of (SE_{n+1}). If E has the Radon-Nikodym property and $n \geq 1$, then v is a strong solution of (SE_n) if and only if v' is a mild solution of (SE_{n-1}).

For every $n \geq 1$ we will write e_n for the function $e_n(\lambda) := \frac{1}{\lambda^{n-1}}$.

THEOREM 1.1. *Let A, B be closed operators on a Banach space E, let $x \in E$ be fixed, $n \geq 1$ and $M, \omega \geq 0$. Then the following statements are equivalent.*

(i) There exists a strong solution v of (SE_n) with $\|v\|^B_{Lip(\omega)} \leq M$.

(ii) There exists $y : (\omega, \infty) \to E$ with $y(\lambda) \in (\lambda B - A)^{-1}x$ for all $\lambda > \omega$, $e_n y \in C^\infty_\omega([D(B)])$, and $\|e_n y\|^B_\omega \leq M$.

(iii) There exists $y : (\omega, \infty) \to E$ with $y(\lambda) \in (\lambda B - A)^{-1}x$ for all $\lambda > \omega$ and $v \in Lip_\omega([D(B)])$ with $\|v\|^B_{Lip(\omega)} \leq M$ such that $y(\lambda) = \lambda^{n-1}\int_0^\infty e^{-\lambda t}\, dv(t)$ for all $\lambda > \omega$.

PROOF. The equivalence of the statements (ii) and (iii) follows from the Widder-Arendt Theorem mentioned before; i.e., from the fact that the Laplace-Stieltjes transform is an isometric isomorphism between $Lip_\omega([D(B)])$ and $C^\infty_\omega([D(B)])$.

We show that (i) implies (iii). If v is a strong solution of (SE_n), then

$$\int_0^\infty e^{-\lambda t}\left(Bv(t) - \frac{t^n}{n!}x\right)\, dt \ = \ \int_0^\infty e^{-\lambda t}\left(A\int_0^t v(s)\, ds\right)\, dt$$

for all $\lambda > \omega$. It follows from the closedness of the operators A and B that $\int_0^\infty e^{-\lambda t} v(t)\, dt \in D(A) \cap D(B)$ and

$$B\int_0^\infty e^{-\lambda t} v(t)\, dt - \frac{1}{\lambda^{n+1}}x \ = \ A\int_0^\infty e^{-\lambda t}\int_0^t v(s)\, ds\, dt \ = \ \frac{1}{\lambda}A\int_0^\infty e^{-\omega t} v(t)\, dt$$

for all $\lambda > \omega$. Hence, $y(\lambda) := \lambda^n \int_0^\infty e^{-\lambda t} v(t)\, dt = \lambda^{n-1}\int_0^\infty e^{-\lambda t}\, dv(t)$ satisfies (iii).

We show that (iii) implies (i). Let $w(t) := \int_0^t v(s)\, ds$. Then $w \in Lip_{\omega'}([D(B)])$ for $\omega' > \omega$. Let Γ be the path $\omega' + 1 + i\mathbb{R}$. It follows with the complex inversion of the Laplace-Stieltjes transform mentioned above that $v(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\frac{y(\lambda)}{\lambda^n}\, d\lambda$ and $w(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\frac{y(\lambda)}{\lambda^{n+1}}\, d\lambda$. By assumption, $Bv \in Lip_\omega(E)$. Hence, by the closedness of B, $Bv(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\frac{By(\lambda)}{\lambda^n}\, d\lambda$. The closedness of A implies that $w(t) \in D(A)$ for all $t \geq 0$ and

$$Aw(t) \ = \ \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\frac{\lambda By(\lambda) - x}{\lambda^{n+1}}\, d\lambda \ = \ Bv(t) - \frac{t^n}{n!}x.$$

$\diamondsuit$

One should notice that except for closedness, no assumptions are made on A, B in the theorem above. As a consequence, only integral solutions are obtained. It can be shown that the time regularity of the solutions improves if additional assumptions are made, i.e. if one assumes that $\lambda B - A$ has a bounded inverse for all $\lambda > \omega$ (see Favini (1979) and Section 2). To obtain mild solutions of (SE) without the existence of a bounded inverse of $\lambda B - A$, it is necessary to make additional assumptions on the Banach space E. The following *local Hille-Yosida Theorem* is an immediate consequence of Theorem 1.1.

COROLLARY 1.2. *Let A, B be closed operators on a Banach space E with the Radon-Nikodym property, let $x \in E$ be fixed, and $M, \omega \geq 0$. Then the following statements are equivalent.*

(i) There exists a mild solution u of (SE) with $\|u\|_{\infty,\omega}^{B} \leq M$.
(ii) There exists $y : (\omega, \infty) \to E$ with $y(\lambda) \in (\lambda B - A)^{-1}x$ for all $\lambda > \omega$, $y \in C_{\omega}^{\infty}([D(B)])$, and $\|y\|_{\omega}^{B} \leq M$.

Assume that the equation $\lambda By - Ay = x$ has a solution $y = y(\lambda)$ which is analytic in a half-plane $H_\omega = \{\lambda \in \mathbb{C} : Re\lambda > \omega \geq 0\}$. In order to apply statement (ii) of Theorem 1.1 one has to show that there exist $n \in \mathbb{N}$ and $M > 0$ such that $\|e_n y\|_{\omega}^{B} \leq M$. Next we will indicate the proof of the fact that if $\|y(\lambda)\|^B \leq C|\lambda|^k$ for all $\lambda \in H_\omega$ and some constants $C > 0$, $k \in \mathbb{N}_0 \cup \{-1\}$, then $\|e_{k+3}y\|_{\omega}^{B} \leq M$, where the constant M depends only on C and ω.

THEOREM 1.3. *Let A, B be closed operators on a Banach space E and let $x \in E$ be fixed. Assume that there exists $\omega \geq 0$ such that the equation $\lambda By - Ay = x$ has a solution $y = y(\lambda)$ for all $\lambda > \omega$. If the function y has an analytic extension into the half-plane $\{\lambda \in \mathbb{C} : Re\lambda > \omega\}$ satisfying there $\|y(\lambda)\|^B \leq C|\lambda|^k$ for some $C > 0$ and $k \in \mathbb{N}_0 \cup \{-1\}$, then there exists a locally Hölder continuous mild integral solution v of (SE_{k+2}) which is $O(e^{\omega' t})$ for all $\omega' > \omega$.*

PROOF. It follows from the analyticity of y and the growth assumptions made that the functions

$$v(t) := \frac{1}{2\pi i}\int_{w+1+i\mathbb{R}} e^{\lambda t}\frac{y(\lambda)}{\lambda^{k+2}}\,d\lambda\ ,\ w(t) := \frac{1}{2\pi i}\int_{w+1+i\mathbb{R}} e^{\lambda t}\frac{By(\lambda)}{\lambda^{k+2}}\,d\lambda$$

are well-defined for all $t > 0$ and $O(e^{w't})$. It follows from the closedness of B that $w(t) = Bv(t)$. It can be shown (see W. Arendt, H. Kellermann (1989), Prop. 3.1) that for all $0 < \epsilon < 1$ and $T > 0$ there exists $C_T > 0$ such that $\|v(t) - v(s)\|^B \leq C_T|t-s|^\epsilon$ for all $t, s \in [0,T]$, that $v(0) = 0$, and that $y(\lambda) = \lambda^{k+2}\int_0^\infty e^{-\lambda t}\,d\alpha(t)$, where $\alpha\ :\ t \to \int_0^t v(s)\,ds \in Lip_\omega([D(B)])$. It follows from Theorem 1.1 that α is a strong solution of (SE_{k+3}). Because of the differentiability of α, the function $v = \alpha'$ is a mild solution of (SE_{k+2}) for all $t \geq 0$. $\diamondsuit$

To obtain integral solutions $v \in Lip_0([0,\infty), E)$ of (SE) which admit an analytic extension into a sector $\Sigma(\beta) := \{0 \neq \lambda \in \mathbb{C} : |arg(\lambda)| < \beta\}$ for some $0 < \beta < \pi/2$, one can apply the following result from Laplace transform theory (see Sova (1979) or Neubrander (1989)).

Let $\omega \in \mathbb{R}$ and $0 < \beta \leq \pi$. Then the closure of the sector $\Sigma(\omega,\beta) := \{\lambda \in \mathbb{C}\backslash\{\omega\} : |arg(\lambda-\omega)| < \beta\}$ will be denoted by $\Sigma'(\omega,\beta)$, and the closure of $\Sigma(\omega,\beta)$ without the point ω by $\Sigma"(\omega,\beta)$.

> Let $\omega \geq 0$ and let r be a function from (ω,∞) into a Banach space E. Then the following statements are equivalent.
>
> (i) There exist $M, \beta > 0$ and $v \in C(\Sigma'(\beta), E)$ which is analytic in $\Sigma(\beta)$, $v(0) = 0$ and $\|v(t+h) - v(t)\| \leq M|h|e^{\omega Re(t+h)}$ for $t, h \in \Sigma'(\beta)$ such that $r(\lambda) = \int_0^\infty e^{-\lambda t}\,dv(t)$ for all $\lambda > \omega$.
>
> (ii) There exist $M, \gamma > 0$ such that r is continuous in $\Sigma"(\omega,\pi/2+\gamma)$, analytic in $\Sigma(\omega,\pi/2+\gamma)$ and satisfies $\|(\lambda-\omega)r(\lambda)\| \leq M$ for all $\lambda \in \Sigma"(\omega,\pi/2+\gamma)$.

THEOREM 1.4. *Let A, B be closed operators on a Banach space E and let $x \in E$ be fixed. Assume that there exists $\omega \geq 0$ such that the equation $\lambda By - Ay = x$ has a solution $y = y(\lambda)$ for all $\lambda > \omega$. If the function $y\ :\ (\omega,\infty) \to [D(B)]$ has an analytic extension into a sector $\Sigma(\omega,\pi/2+\gamma)$ for some $\gamma > 0$ such that $\|(\lambda-\omega)y(\lambda)\|^B \leq M|\lambda|^k$ for some constants $M > 0$, $k \in \mathbb{N}_0$ and all $\lambda \in \Sigma''(\omega,\pi/2+\gamma)$, then there exists a strong solution $v \in Lip_\omega([D(B)])$ of (SE_{k+1}) which is analytic in a sector $\Sigma(\beta)$ for some $\beta > 0$. Moreover, the function $v^{(k+1)}$ solves $\frac{d}{dt}Bu(t) = Bu'(t) = Au(t)$ for all $t > 0$.*

PROOF. The statement follows immediately from the afore mentioned result from Laplace transform theory and Theorem 1.1. $\diamondsuit$

So far we have seen that for the existence of a strong exponentially bounded solution of (SE_n) for some $n \in \mathbb{N}$ it is necesssary and sufficient to assume that there exists $\omega \geq 0$ such that

(a) $x \in \bigcap_{\lambda > \omega} Range(\lambda B - A)$,
(b) there exists a function $y : (\omega, \infty) \to [D(B)]$ with $y(\lambda) \in (\lambda B - A)^{-1}x$ for all $\lambda > \omega$ which has an analytic extension into the right half-plane H_w such that $\|y(\lambda)\|^B \leq p(|\lambda|)$ for some polynomial p and all $\lambda \in H_\omega$.

If the operators A, B and $x \in E$ satisfy the range condition (a) and the growth condition (b) above, then there exists a strong integral solution v of (SE) which is given by the inverse Laplace-Stieltjes transform L_S^{-1} of the function $\lambda \to y(\lambda)/\lambda^n$ for some large enough $n \in \mathbb{N}$. In order to obtain uniqueness of exponentially bounded integral solutions a condition on the point spectrum $p\sigma(A, B) := \{\lambda : (\lambda B - A)x = 0 \text{ for some } 0 \neq x \in D(A) \cap D(B)\}$ is needed.

For the following result the Uniqueness Theorem for Laplace-Stieltjes transforms of functions $v \in Lip_\omega(E)$ is needed; i.e., if there are $\lambda_0, \lambda_1 > 0$ such that $r(\lambda_n) = \int_0^\infty e^{-\lambda_n t}\, dv(t) = 0$ for all $\lambda_n = \lambda_0 + n\lambda_1$, $(n \in \mathbb{N}_0)$, then $v = 0$ (see, for example, Hennig and Neubrander (1990) or Neubrander (1992)).

THEOREM 1.5. *Let A, B be closed operators on a Banach space E and $n \in \mathbb{N}_0$. If there exist $\lambda_0, \lambda_1 > 0$ such that the equidistant points $\lambda_k = \lambda_0 + k\lambda_1$, $k \in \mathbb{N}_0$ do not belong to the point spectrum $p\sigma(A, B)$, then (SE_n) has at most one exponentially bounded mild solution.*

PROOF. Let v be a mild solution of (SE_n) for $x = 0$. Then $\alpha(t) := \int_0^t v(s)\, ds$ is a strong solution of (SE_k) for all $k \geq 0$. Let $y := L_S(\alpha)$. By Theorem 1.1, there exists $\omega \geq 0$ such that $\lambda By(\lambda) - Ay(\lambda) = 0$ for all $\lambda > \omega$. Because the points λ_k do not belong to $p\sigma(A, B)$, it follows that $y(\lambda_k) = 0$ for all $\lambda_k > w$. By the Uniqueness Theorem for Laplace-Stieltjes transforms, $\alpha = 0$. $\diamondsuit$

2. CLASSICAL SOLUTIONS

Let A, B be closed operators in E such that the operators $\lambda B - A$ with domains $D(A) \cap D(B)$ are closed and have a bounded inverse $R(\lambda) := (\lambda B - A)^{-1}$ for all $\lambda > \omega \geq 0$. Then $R(\lambda)$ is in $\mathcal{L}(E, [D(B)])$, the space of all bounded operators from E into $[D(B)]$. Because $\lambda B R(\lambda) = I + AR(\lambda)$ and $R(\lambda)Ax = \lambda R(\lambda)Bx - x$ for all $x \in D(A) \cap D(B)$, it follows that $\lambda BR(\lambda)Ax = Ax + AR(\lambda)Ax = Ax + [\lambda R(\lambda)Bx - x] = \lambda AR(\lambda)Bx$ and therefore

$$BR(\lambda)Ax = AR(\lambda)Bx \tag{2.1}$$

for all $x \in D(A) \cap D(B)$. Assume further that $R(\lambda)$ and $BR(\lambda)$ have an analytic extension into the half-plane $H_\omega := \{\lambda : Re\lambda > w\}$ and that there exists a polynomial p such that

$$\|R(\lambda)\| + \|BR(\lambda)\| \leq p(|\lambda|) \tag{2.2}$$

for all $\lambda \in H_\omega$. It follows from Theorem 1.3 that the growth condition (2.2) holds in some half-plane H_ω if and only if there exist $n \in \mathbb{N}_0$ and a strongly continuous operator family $(S(t))_{t \geq 0} \subset \mathcal{L}(E, [D(B)])$ with $\|S(t)x\|^B \leq Me^{\omega t}\|x\|$ $(t \geq 0)$ and

$$R(\lambda)x = \lambda^n \int_0^\infty e^{-\lambda t} S(t)x\, dt \tag{2.3}$$

for some constants $M, \omega \geq 0$, all $x \in E$ and all $\lambda > \omega$. If (2.3) holds, then we say that A, B generate the n-times integrated evolution family $S(t)$.

PROPOSITION 2.1. *Let A, B generate an n-times integrated evolution family $S(t)$. Then*

(i) $S(t)Bx = \frac{t^n}{n!}x + \int_0^t S(s)Ax\, ds$ *for all* $x \in D(A) \cap D(B)$.
(ii) $\int_0^t S(s)x\, ds \in D(A) \cap D(B)$ *for all* $x \in E$.
(iii) $BS(t)x = \frac{t^n}{n!}x + A\int_0^t S(s)x\, ds$ *for all* $x \in E$.
(iv) $S(t)B : D(A) \cap D(B) \to D(A) \cap D(B)$.
(v) $AS(t)Bx = BS(t)Ax$ *for all* $x \in D(A) \cap D(B)$.

PROOF. If (2.3) holds, then the assumed continuity of $t \to BS(t)x$ and the closedness of B implies that $BR(\lambda)x = \lambda^n \int_0^\infty e^{-\lambda t} BS(t)x\, dt$ for all $x \in E$ and $\lambda > w$. It follows from $\frac{x}{\lambda^{n+1}} = \frac{R(\lambda)Bx}{\lambda^n} - \frac{R(\lambda)Ax}{\lambda^{n+1}}$ for all $x \in D(A) \cap D(B)$ that

$$\int_0^\infty e^{-\lambda t} \frac{t^n}{n!} x\, dt = \int_0^\infty e^{-\lambda t} S(t)Bx\, dt - \int_0^\infty e^{-\lambda t} \int_0^t S(s)Ax\, ds\, dt.$$

The Uniqueness Theorem for Laplace transforms yields statement (i).

Define $S^{[1]}(t) : x \to \int_0^t S(s)x\, ds$. Then $S^{[1]}(\cdot)x \in Lip_\omega([D(B)])$, $\|S^{[1]}(\cdot)x\|^B_{Lip(\omega)} \le M\|x\|$ and $R(\lambda)x = \lambda^n \int_0^\infty e^{-\lambda t}\, dS^{[1]}(t)x$ for all $x \in E$ and $\lambda > w$. As shown in Theorem 1.1, $\int_0^t S^{[1]}(s)x\, ds \in D(A)$ and

$$A\int_0^t S^{[1]}(s)x\, ds = BS^{[1]}(t)x - \frac{t^{n+1}}{(n+1)!}x = \int_0^t BS(s)x\, ds - \frac{t^{n+1}}{(n+1)!}x.$$

It follows from the closedness of A that $S^{[1]}(t)x \in D(A)$ and

$$AS^{[1]}(t)x = BS(t)x - \frac{t^n}{n!}x$$

for all $x \in E$ and $t \ge 0$. This implies the statements (ii) and (iii).

The statement (iv) follows from (i) and (ii). Let $x \in D(A) \cap D(B)$. Then (i) and (iii) imply that $AS(t)Bx = \frac{t^n}{n!}Ax + A\int_0^t S(s)Ax\, ds = BS(t)Ax$. This shows (v). ◊

In the proof of Theorem 2.3 the following lemma is needed.

LEMMA 2.2. *Let A, B generate an n-times integrated evolution family $S(t)$, let $j \in \mathbb{N}$, and $x \in D(A) \cap D(B)$. Then $\frac{d}{dt}(\lambda_0 R(\lambda_0)B - I)^j S(t)Bx = (\lambda_0 R(\lambda_0)B - I)^j \frac{d}{dt} S(t)Bx$.*

PROOF. The statement is proved by induction. Let $j = 1$. Then $(\lambda_0 R(\lambda_0)B - I)S(t)Bx = \lambda_0 R(\lambda_0)BS(t)Bx - S(t)Bx$. It follows from Proposition 2.1 (i) and $BS(t)Bx = \frac{t^n}{n!}Bx + \int_0^t BS(s)Ax\, ds$ for all $x \in D(A) \cap D(B)$ that

$$\frac{d}{dt}S(t)Bx = \frac{t^{n-1}}{(n-1)!}x + S(t)Ax \tag{2.4}$$

and $\frac{d}{dt}BS(t)Bx = B\frac{d}{dt}S(t)Bx$. Hence, $\frac{d}{dt}(\lambda_0 R(\lambda_0)B - I)S(t)Bx = \lambda_0 R(\lambda_0)\frac{d}{dt}BS(t)Bx - \frac{d}{dt}S(t)Bx = (\lambda_0 R(\lambda_0)B - I)\frac{d}{dt}S(t)Bx$.

Assume that the statement is true for some $j \in \mathbb{N}$. Because $(\lambda_0 R(\lambda_0)B - I)^j R(\lambda_0) \in \mathcal{L}(E)$ it follows that $\frac{d}{dt}(\lambda_0 R(\lambda_0)B - I)^{j+1} S(t)Bx = \frac{d}{dt}(\lambda_0 R(\lambda_0)B - I)^j \lambda_0 R(\lambda_0)BS(t)Bx - \frac{d}{dt}(\lambda_0 R(\lambda_0)B - I)^j S(t)Bx = (\lambda_0 R(\lambda_0)B - I)^j \lambda_0 R(\lambda_0)B\frac{d}{dt}S(t)Bx -$
$(\lambda_0 R(\lambda_0)B - I)^j \frac{d}{dt}S(t)Bx = (\lambda_0 R(\lambda_0)B - I)^{j+1}\frac{d}{dt}S(t)Bx$. ◊

THEOREM 2.3. *Let A, B generate an n-times integrated evolution family $S(t)$ and let $\lambda_0 > w$. Then*

$$Bu'(t) = Au(t)\ ;\ u(0) = (R(\lambda_0)B)^n\, x \qquad (ACP_B)$$

has a unique solution $u \in C^1([0,\infty), E]) \cap C([0,\infty), [D(B)]) \cap C([0,\infty), [D(A)])$ for all $x \in D(A) \cap D(B) \cap D\left((R(\lambda_0)B)^n\right)$.

PROOF. Define

$$u(t) := \sum_{j=0}^{n-1} \frac{t^j}{j!}(R(\lambda_0)B)^{n-j}(\lambda_0 R(\lambda_0)B - I)^j x \ + \ (\lambda_0 R(\lambda_0)B - I)^n S(t)Bx.$$

If $n = 0$ and $k = 0, 1$, then we set $\sum_{j=k}^{-1} \cdot \ := 0$. Clearly, $u(0) = (R(\lambda_0)B)^n x$. By Proposition 2.1, $u \in C([0,\infty), [D(B)]) \cap C([0,\infty), [D(A)])$. It follows from Lemma 2.2 and (2.4) that $u \in C^1([0,\infty), E)$ and

$$\begin{aligned} u'(t) &= \sum_{j=1}^{n-1} \frac{t^{j-1}}{(j-1)!}(R(\lambda_0)B)^{n-j}(\lambda_0 R(\lambda_0)B - I)^j x \ + \ (\lambda_0 R(\lambda_0)B - I)^n \frac{d}{dt} S(t)Bx \\ &= \sum_{j=0}^{n-2} \frac{t^j}{j!}(R(\lambda_0)B)^{n-j-1}(R(\lambda_0)A)^{j+1} x \ + \ \frac{t^{n-1}}{(n-1)!}(R(\lambda_0)A)^n x \\ &\quad + \ (\lambda_0 R(\lambda_0)B - I)^n S(t)Ax \ \in \ D(B) \end{aligned}$$

for all $t \geq 0$. Proposition 2.1 (v) and (2.1) yields

$$\begin{aligned} Bu'(t) &= \sum_{j=0}^{n-2} \frac{t^j}{j!} B(R(\lambda_0)B)^{n-j-1}(R(\lambda_0)A)^{j+1} x \ + \ \frac{t^{n-1}}{(n-1)!} B(R(\lambda_0)A)^n x \\ &\quad + \ B(\lambda_0 R(\lambda_0)B - I)^n S(t)Ax \\ &= A \sum_{j=0}^{n-2} \frac{t^j}{j!}(R(\lambda_0)B)^{n-j}(R(\lambda_0)A)^j x \ + \ A\frac{t^{n-1}}{(n-1)!}(R(\lambda_0)B)(R(\lambda_0)A)^{n-1} x \\ &\quad + \ A(\lambda_0 R(\lambda_0)B - I)^n S(t)Bx \\ &= Au(t). \end{aligned}$$

The uniqueness of the solution u follows from Theorem 3.1 in Favini (1979). ◊

REMARKS 2.4. (1) If $B, B^{-1} \in \mathcal{L}(E)$, then $B^{-1}A$ with domain $D(A)$ is a closed operator. If A, B generate an n-times integrated evolution family $S(t)$, then $B^{-1}A$ generates the n-times integrated semigroup $BS(t)$. In this case Theorem 2.3 restates the well known fact that for such operators A, B the abstract Cauchy problem $u'(t) = B^{-1}Au(t)$, $u(0) = z$ has a unique solution for all $z \in D((B^{-1}A)^{n+1})$ (see, for example, Arendt (1987), Neubrander (1988), or Hieber, Holderrieth and Neubrander (1991)).

(2) Let $A^{-1} \in \mathcal{L}(E)$.If there exist $k \in \mathbb{N}_0$ and $C, \omega \geq 0$ such that $\|AR(\lambda)\| \leq C|\lambda|^k$ for all $\lambda \in \mathbb{C}$ with $Re\lambda > \omega$, then $\|R(\lambda)\|^B \leq M\|\lambda|^k$ for some constant M and all $\lambda \in \mathbb{C}$ with $Re\lambda > \omega' := max(\omega, 1)$. By Theorem 1.3, $R(\lambda)x = \lambda^{k+2}\int_0^\infty e^{-\lambda t}S(t)x\,dt$ for all $\lambda > \omega'$, where $S(t)x = v(t)$, and $v(t)$ as in the proof of Theorem 1.3. It follows from Proposition 2.1 (iii) that $\frac{d}{dt}A^{-1}BS(t)x = \frac{t^{n-1}}{(n-1)!}A^{-1}x + S(t)x$ for all $x \in E$. For $x \in D((A^{-1}B)^{k+2}A^{-1})$ define

$$w(t) := \sum_{i=0}^{k+1} \frac{t^i}{i!}(A^{-1}B)^{k+2-i}A^{-1}x + A^{-1}BS(t)x.$$

Then $Bw'(t) = Aw(t)$, $w(0) = (A^{-1}B)^{k+2}A^{-1}x$. This proves Theorem 4.2 in Favini (1979).

(3) Let $0 \notin p\sigma(B)$ and A, B generate an n-times integrated evolution family $S(t)$. It follows from Proposition 2.1 (i) that $\frac{d}{dt}S(t)x = \frac{t^{n-1}}{(n-1)!}B^{-1}x + S(t)AB^{-1}x$ for all $x \in D(AB^{-1}) = \{x \in D(B^{-1}) : B^{-1}x \in D(A)\}$. For $x \in D((AB^{-1})^{n+1})$ define

$$w(t) := \sum_{i=0}^{n-1} \frac{t^i}{i!}B^{-1}(AB^{-1})^iA^{-1}x + S(t)(AB^{-1})^nx.$$

Then $w'(t) = B^{-1}Aw(t)$, $w(0) = B^{-1}x$.

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